3 – IMAGE TRANSFORMS (A)

The Fourier transform decomposes an image into its sinusoidal components, thus enabling easy examination and processing of certain frequencies of the image. It is used in algorithms for analysis, filtering, reconstruction, compression and restoration.

MATHEMATICAL PRELIMINARIES

Dirac Delta Function

The 2D Dirac delta function, $\delta(x, y)$ is a singularity operator that is defined by:

$$\iint_{-\infty}^{+\infty} \delta(x,y)dxdy = 1 \tag{1}$$

$$\delta(x,y) = \begin{cases} 0 & x,y \neq 0 \\ \infty & x,y = 0 \end{cases} \tag{2}$$

Useful properties are

$$\delta(x,y) = \delta(x)\delta(y) \tag{3}$$

$$\iint_{-\infty}^{+\infty} f(x,y)\delta(x,y)dxdy = f(0,0) \tag{4}$$

$$\iint_{-\infty}^{+\infty} f(x,y)\delta(x-\alpha, y-\beta)dxdy = f(\alpha,\beta) \qquad \text{(Sifting Property)} \tag{5}$$

$$\iint_{-\infty}^{+\infty} \exp[j2\pi(ux+vy)]dudv = \delta(x,y) \tag{6}$$

System Response

The output of a system with $\delta(x, y)$ as the input function is termed the impulse response, i.e.,

$$h(x, y) \equiv \mathcal{O}\{\delta(x, y)\} \tag{7}$$

In optical systems, the impulse response is often called the point spread function (PSF).

The response of a system to an arbitrary input $f(x, y)$ is found by convolving $h(x, y)$ with $f(x, y)$:

$$\begin{aligned} g(x, y) &= h(x, y) \star f(x, y) & (8) \\ &= \iint_{-\infty}^{+\infty} h(\alpha, \beta) f(x - \alpha, y - \beta) d\alpha \, d\beta & (9) \end{aligned}$$

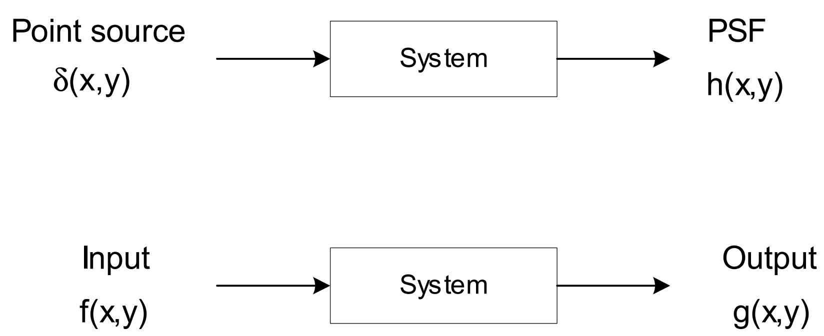
It can easily be shown that

$$h(x, y) \star f(x, y) = f(x, y) \star h(x, y) \tag{10}$$

The system transfer function is

$$H(u, v) \equiv \mathcal{F}\{h(x, y)\} \tag{11}$$

where $\mathcal{F}$ is the Fourier transform operator. In the case of optical systems, the transfer function is called the *optical transfer function* (OTF).



THE FOURIER TRANSFORM

The 2D Fourier transform of the function $f(x,y)$ is defined as

$$F(u,v) = \int\int_{-\infty}^{+\infty} f(x,y)\exp[-j2\pi(ux+vy)]dxdy \tag{12}$$

and its inverse is defined as

$$f(x,y) = \int\int_{-\infty}^{+\infty} F(u,v)\exp[j2\pi(ux+vy)]dudv \tag{13}$$

$F(u,v)$ is called the frequency spectrum of $f(x,y)$. If x and y represent spatial coordinates, then u and v are the spatial frequencies (measured in cycles per unit distance) along the x and y axes, respectively.

The Fourier transform of f may not exist unless f satisfies certain conditions. The following is a typical set of sufficient conditions for its existence:

1. $\int\int_{-\infty}^{+\infty} |f(x,y)|dxdy < \infty$
2. $f(x,y)$ must have only a finite number of discontinuities and a finite number of maxima and minima in any finite rectangle.
3. $f(x,y)$ must have no infinite discontinuities.

The 2D Fourier transform can be computed in two separable steps:

$$F(u,v) = \int_{-\infty}^{+\infty} \left[\int_{-\infty}^{+\infty} f(x,y) \exp(-j2\pi ux)dx \right] \exp(-j2\pi vy)dy \quad (14)$$

$$= \int_{-\infty}^{\infty} F(u,y) \exp(-j2\pi vy)dy \quad (15)$$

where

$$F(u,y) = \int_{-\infty}^{\infty} f(x,y) \exp(-j2\pi ux)dx \quad (16)$$

In general, the Fourier transform is complex:

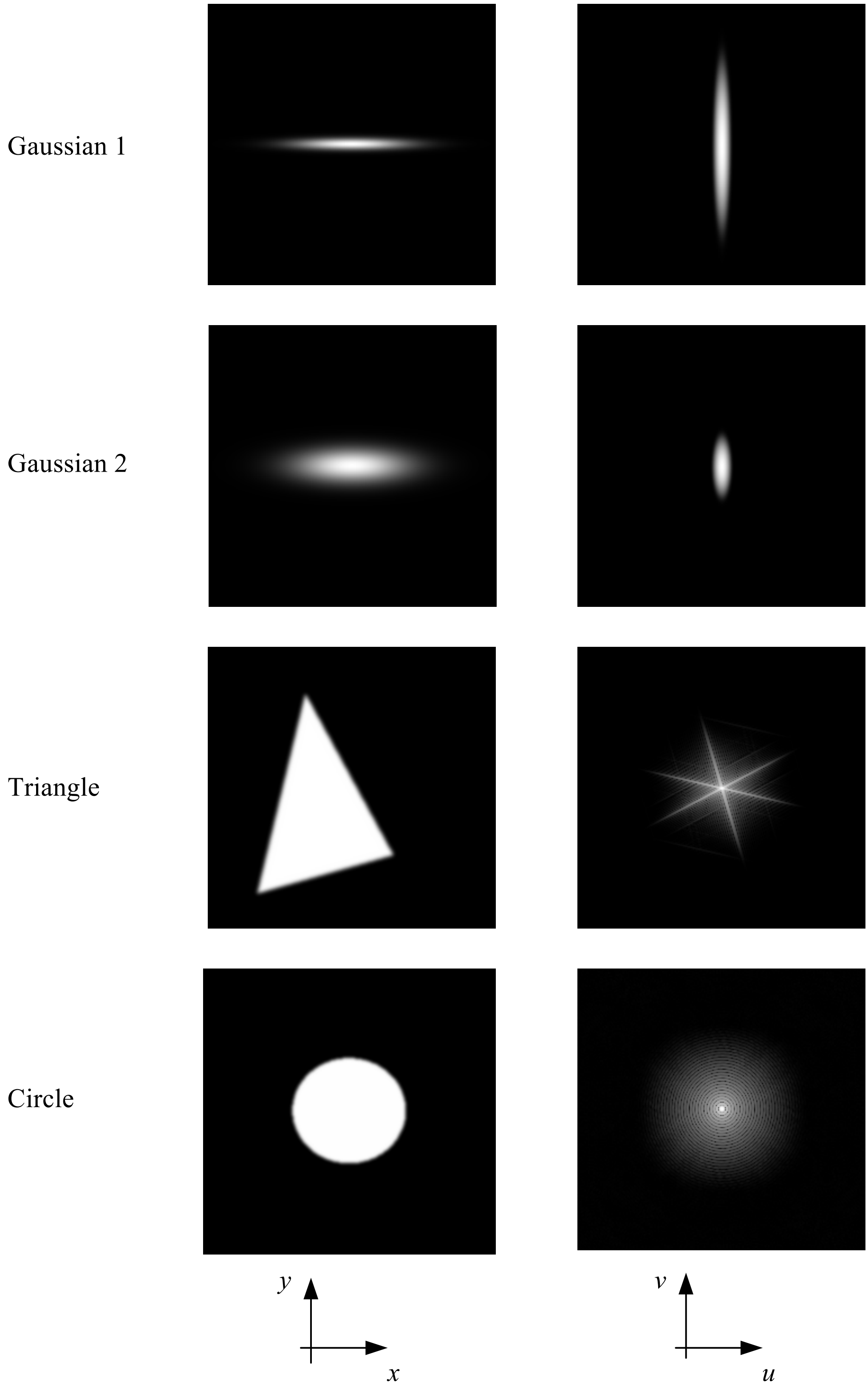
$$F(u,v) \equiv R(u,v) + jI(u,v) \equiv |F(u,v)| \exp[j\phi(u,v)]$$

with real part $R(u,v)$ and an imaginary part $I(u,v)$: We define

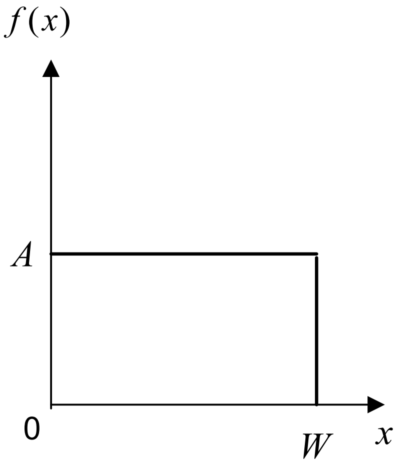
Fourier spectrum:	$\|F(u,v)\|$	$= [R^2(u,v) + I^2(u,v)]^{1/2}$
Phase spectrum:	$\phi(u,v)$	$= \tan^{-1}[I(u,v)/R(u,v)]$
Power spectrum:	$P(u,v)$	$= \|F(u,v)\|^2 = R^2(u,v) + I^2(u,v)$

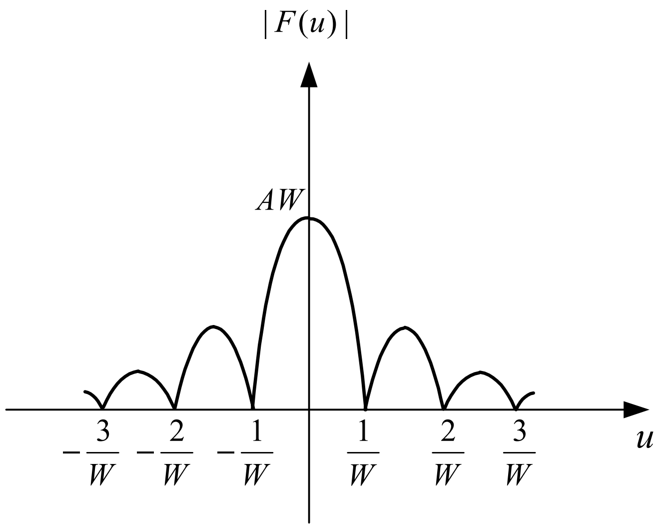
Fourier transform examples

Images and their transform magnitudes (log scale)



Example



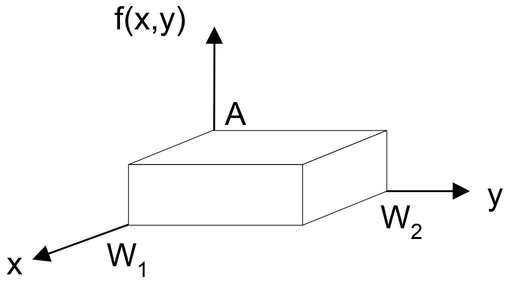


The Fourier transform of $f(x)$ is

$$
\begin{aligned}
F(u) &= \int_{-\infty}^{\infty} f(x)\exp(-j2\pi ux)dx \\
&= A\int_{0}^{W} \exp(-j2\pi ux)dx \\
&= \frac{-A}{j2\pi u}[\exp(-j2\pi ux)]_0^W \\
&= \frac{-A}{j2\pi u}[\exp(-j2\pi uW) - 1] \\
&= \frac{-A}{j2\pi u}\exp(-j\pi uW)[\exp(-j\pi uW) - \exp(j\pi uW)] \\
&= \frac{-A}{j2\pi u}\exp(-j\pi uW)[-2j\sin(\pi uW)] \\
&= A\frac{\sin(\pi uW)}{\pi u}\exp(-j\pi uW) \\
&= AW\frac{\sin(\pi uW)}{\pi uW}\exp(-j\pi uW) \\
&= AW\operatorname{sinc}(uW)\exp(-j\pi uW)
\end{aligned}
$$

$$
|F(u)| = AW|\operatorname{sinc}(uW)|
$$

Example

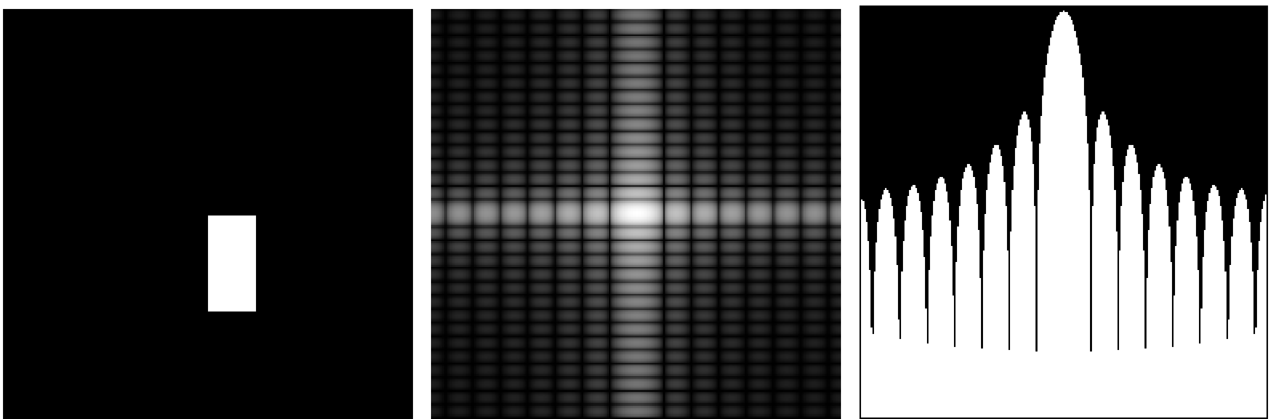


The Fourier transform of $f(x,y)$ is

$$
\begin{aligned}
F(u,v) &= \int\int_{-\infty}^{+\infty} f(x,y)\exp[-j2\pi(ux+vy)]dxdy \\
&= A\int_0^{W_1}\exp[-j2\pi ux]dx\int_0^{W_2}\exp[-j2\pi vy]dy \\
&= A\left[\frac{\exp(-j2\pi ux)}{-j2\pi u}\right]_0^{W_1}\left[\frac{\exp(-j2\pi vy)}{-j2\pi v}\right]_0^{W_2} \\
&= AW_1W_2\mathrm{sinc}(uW_1)\mathrm{sinc}(vW_2)\exp[-j\pi(uW_1+vW_2)]
\end{aligned}
$$

The spectrum is

$$|F(u,v)| = AW_1W_2|\mathrm{sinc}(uW_1)||\mathrm{sinc}(vW_2)|$$



$f(x,y)$ | $|F(u,v)|$ | Cross-section through centre

Properties of the Fourier Transform

Functional properties

(1) If the image function is spatially separable such that

$$f(x, y) = f_1(x) f_2(y) \tag{17}$$

then

$$F(u, v) = F_1(u) F_2(v) \tag{18}$$

where $F_1(u)$ and $F_2(v)$ are the 1D Fourier transforms of $f_1(x)$ and $f_2(y)$, respectively.

Proof

$$\begin{aligned} f(x, y) &= f_1(x) f_2(y) \\ &\leftrightarrow \int\int f_1(x) f_2(y) \exp[-j2\pi(ux + vy)] dx dy \\ &= \int f_1(x) \exp(-j2\pi ux) dx \int f_2(y) \exp(-j2\pi vy) dy \\ &= F_1(u) F_2(v) \end{aligned}$$

(2) Complex conjugation:

$$\mathcal{F}\{f^*(x,y)\} = F^*(-u,-v) \tag{19}$$

where * denotes the complex conjugation of a variable and $\mathcal{F}$ is the Fourier transform operator.

Proof (1D case)

$$\begin{aligned} F(u) &= \int f(x)\exp(-j2\pi ux)dx \\ F^*(u) &= \int f^*(x)\exp(j2\pi ux)dx \\ F^*(-u) &= \int f^*(x)\exp(-j2\pi ux)dx \\ &= \mathcal{F}\{f^*(x)\} \end{aligned}$$

i.e.,

$$f^*(x) \leftrightarrow F^*(-u) \tag{20}$$

We note that for real $f(x)$,

$$f^*(x) = f(x)$$

i.e.

$$f(x) \leftrightarrow F^*(-u) \tag{21}$$

or

$$\begin{aligned} F(u) &= F^*(-u) \\ |F(u)| &= |F^*(-u)| \\ &= |F(-u)| \end{aligned}$$

$\Rightarrow |F(u)|$ is an even function.

In the 2D case,

$$|F(u,v)| = |F(-u,-v)| \tag{22}$$

i.e., $|F(u,v)|$ is symmetrical about the origin.

(3) Reflection about the origin:

$$\mathcal{F}\{f(-x,-y)\} = F(-u,-v) \tag{23}$$

Proof (1D case)

$$\begin{aligned} \mathcal{F}\{f(-x)\} &= \int f(-x)\exp(-j2\pi ux)dx \\ &= \int f(\alpha)\exp(j2\pi u\alpha)d\alpha \quad \text{by substituting } -x=\alpha \\ &= F(-u) \end{aligned}$$

Linearity

The Fourier transform is a linear operator:

$$\mathcal{F}\{af_1(x,y) + bf_2(x,y)\} = aF_1(u,v) + bF_2(u,v) \tag{24}$$

where a and b are constants.

Scaling

A linear scaling of the spatial variables results in an inverse scaling of the spatial frequencies:

$$\mathcal{F}\{f(ax,by)\} = \frac{1}{|ab|}F(u/a, v/b) \tag{25}$$

Proof (1D case)

$$\begin{aligned} f(ax) &\leftrightarrow \int_{-\infty}^{\infty} f(ax)\exp(-j2\pi ux)dx \qquad a>0 \\ &= \frac{1}{a}\int_{-\infty}^{\infty} f(\lambda)\exp(-j2\pi u\lambda/a)d\lambda \quad \text{by substituting } \lambda = ax \\ &= \frac{1}{a}F(u/a) \end{aligned}$$

$$
\begin{aligned}
f(ax) &\leftrightarrow \int_{-\infty}^{\infty} f(ax)\exp(-j2\pi ux)dx \qquad a<0 \\
&= -\frac{1}{a}\int_{-\infty}^{\infty} f(\lambda)\exp(-j2\pi u\lambda/a)d\lambda \quad \text{where } \lambda = ax \\
&= -\frac{1}{a}F(u/a)
\end{aligned}
$$

Hence

$$\mathcal{F}\{f(ax)\} = \frac{1}{|a|}F(u/a) \tag{26}$$

[**Example**]

Shift

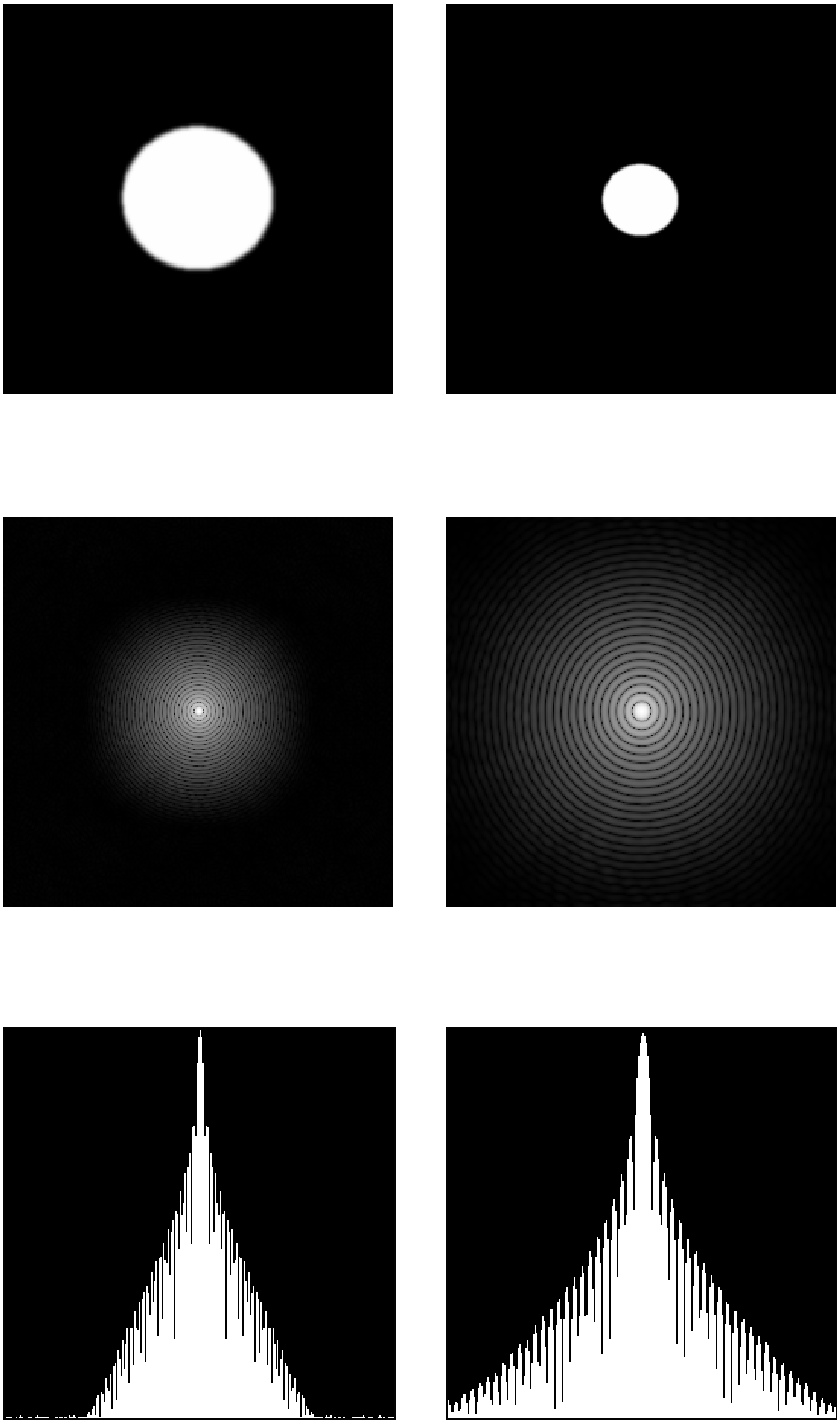
A spatial shift in the input plane results in a phase shift in the output plane:

$$\mathcal{F}\{f(x-a, y-b)\} = F(u,v)\exp[-j2\pi(au+bv)]$$

Proof (1D case)

$$
\begin{aligned}
\mathcal{F}\{f(x-a)\} &= \int f(x-a)\exp(-j2\pi ux)dx \\
&= \int f(\lambda)\exp(-j2\pi u(\lambda+a))dx \quad \text{by substituting } \lambda = x-a \\
&= \exp(-j2\pi au)\int f(\lambda)\exp(-j2\pi u\lambda)d\lambda \\
&= F(u)\exp(-j2\pi au)
\end{aligned}
$$

Scaling



Convolution

The 2D Fourier transform of two convolved functions is equal to the products of the transforms of the functions:

$$\mathcal{F}\{f(x,y) \star h(x,y)\} = F(u,v)H(u,v)$$

Conversely,

$$\mathcal{F}\{f(x,y)h(x,y)\} = F(u,v) \star H(u,v)$$

Parseval's Theorem

The energy in the spatial and Fourier transform domains is related by

$$\int\!\!\int_{-\infty}^{+\infty} |f(x,y)|^2 dx dy = \int\!\!\int_{-\infty}^{+\infty} |F(u,v)|^2 du dv$$

Note that $|F(u,v)|^2$ is termed the energy spectral density.

Proof (1D case)

$$\begin{aligned}
\int |f(x)|^2 dx &= \int f(x) f^*(x) dx \\
&= \int f(x) [\int F^*(u) \exp(-j2\pi ux) du] dx \\
&= \int F^*(u) [\int f(x) \exp(-j2\pi ux) dx] du \\
&= \int F^*(u) F(u) du \\
&= \int |F(u)|^2 du
\end{aligned}$$

Spatial Differentials

We have

$$f(x,y) = \int\int_{-\infty}^{+\infty} F(u,v)\exp[j2\pi(ux+vy)]dudv$$

Differentiating with respect to x,

$$\begin{aligned}\frac{\partial f(x,y)}{\partial x} &= \int\int_{-\infty}^{+\infty} j2\pi u F(u,v)\exp[j2\pi(ux+vy)]dudv \\ &= \mathcal{F}^{-1}\{j2\pi u F(u,v)\}\end{aligned}$$

Therefore,

$$\mathcal{F}\left\{\frac{\partial f(x,y)}{\partial x}\right\} = j2\pi u F(u,v)$$

Similarly,

$$\mathcal{F}\left\{\frac{\partial f(x,y)}{\partial y}\right\} = j2\pi v F(u,v)$$

The Laplacian of a two-variable function $f(x,y)$ is defined as

$$\nabla^2 f(x,y) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$$

Consequently, the Fourier transform of the Laplacian of an image function is

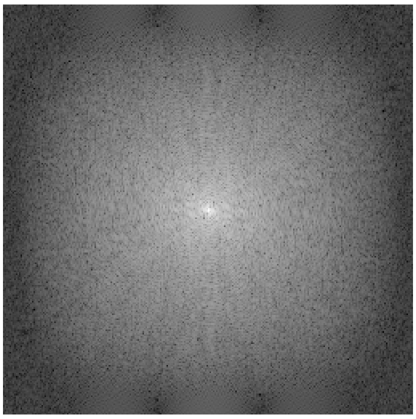
$$\mathcal{F}\left\{\nabla^2 f(x,y)\right\} = -4\pi^2(u^2+v^2)F(u,v)$$

Example

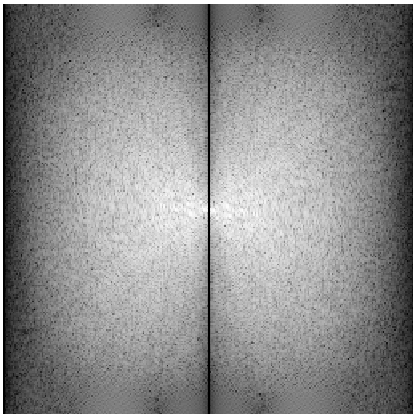
Differential in x (horizontal) direction

CT image

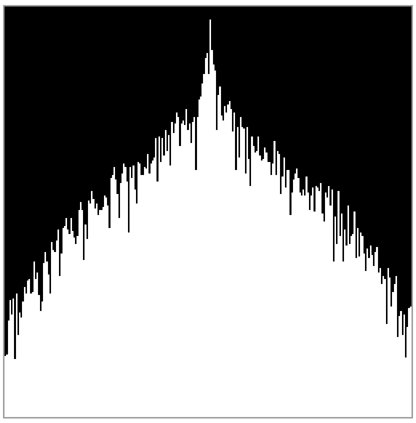
After differentiation (offset by 128)



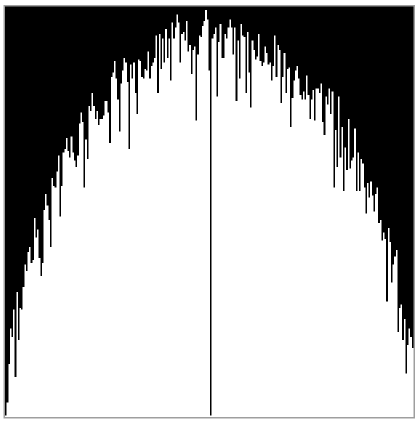
Fourier transform



Fourier transform



Profile



Profile

Rotation

If we introduce the polar coordinates

$$x = r\cos\theta \quad y = r\sin\theta \quad u = \omega\cos\phi \quad v = \omega\sin\phi$$

then $f(x, y)$ and $F(u, v)$ become $f(r, \theta)$ and $F(\omega, \phi)$, respectively. Direct substitution in the continuous Fourier transform pair yields

$$f(r, \theta + \theta_0) \Leftrightarrow F(\omega, \phi + \theta_0) \tag{27}$$

In other words, rotating $f(x, y)$ by an angle θ_0 rotates $F(u, v)$ by the same angle.

Proof

Convert to polar coordinates:

$$\begin{array}{ll} x = r\cos\theta & y = r\sin\theta \\ u = \omega\cos\phi & v = \omega\sin\phi \end{array} \qquad dx\,dy = r dr\,d\theta$$

$$\begin{aligned} F(\omega, \phi) &= \mathcal{F}\{f(r, \theta)\} \\ &= \int_0^{2\pi}\int_0^{\infty} r f(r, \theta)\exp[-j2\pi\omega r(\cos\theta\cos\phi + \sin\theta\sin\phi]dr\,d\theta \\ &= \int_0^{2\pi}\int_0^{\infty} r f(r, \theta)\exp[-j2\pi\omega r\cos(\theta - \phi)]dr\,d\theta \end{aligned}$$

Rotate $f(r, \theta)$ by θ_0:

$$\begin{aligned} F'(\omega, \phi) &= \mathcal{F}\{f(r, \theta + \theta_0)\} \\ &= \int_0^{2\pi}\int_0^{\infty} r f(r, \theta + \theta_0)\exp[-j2\pi\omega r\cos(\theta - \phi)]dr\,d\theta \end{aligned}$$

Let $\lambda = \theta + \theta_0$

$$\begin{aligned} F'(\omega, \phi) &= \int_{\theta_0}^{2\pi+\theta_0}\int_0^{\infty} r f(r, \lambda)\exp[-j2\pi\omega r\cos(\lambda - \theta_0 - \phi)]dr\,d\lambda \\ &= \int_0^{2\pi}\int_0^{\infty} r f(r, \lambda)\exp[-j2\pi\omega r\cos(\lambda - (\theta_0 + \phi))]dr\,d\lambda \end{aligned}$$

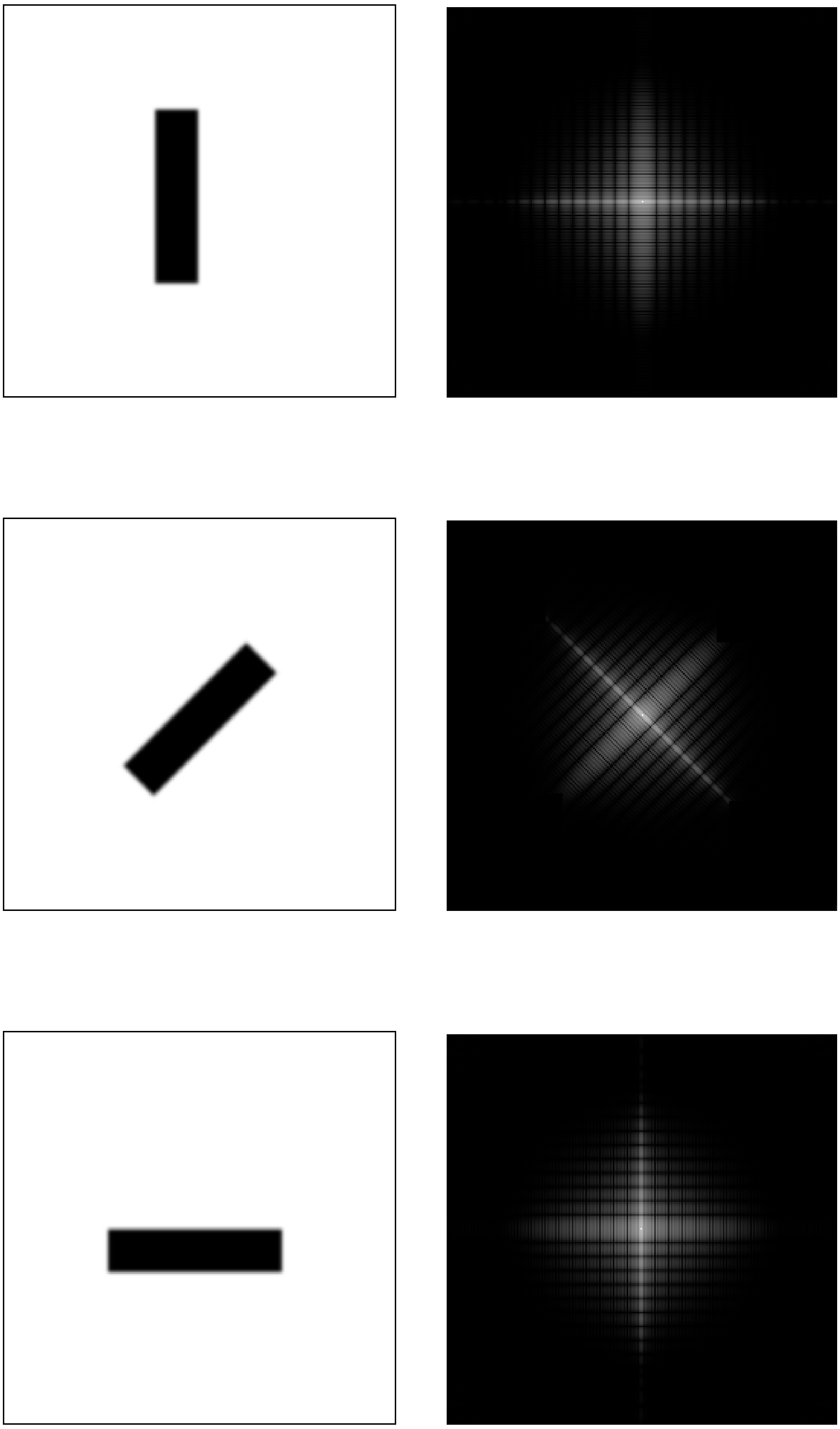
i.e.,

$$F'(\omega, \phi) = F(\omega, \phi + \theta_0)$$

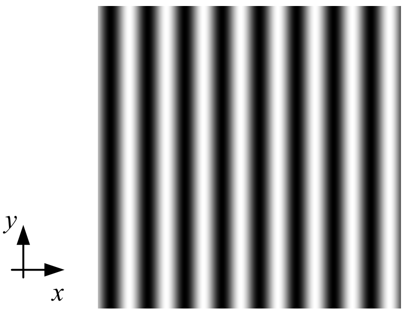
or

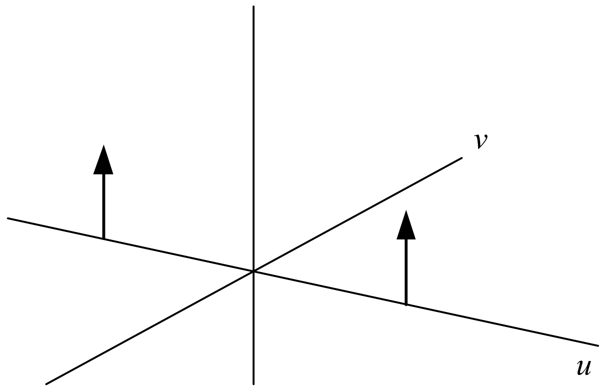
$$f(r, \theta + \theta_0) \leftrightarrow F(\omega, \phi + \theta_0)$$

Example

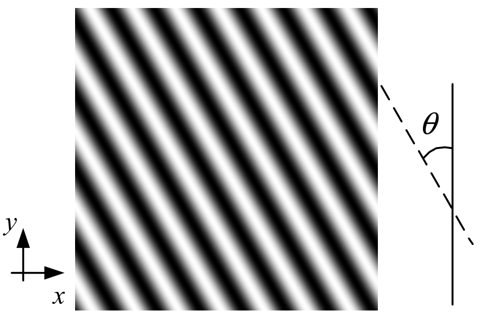


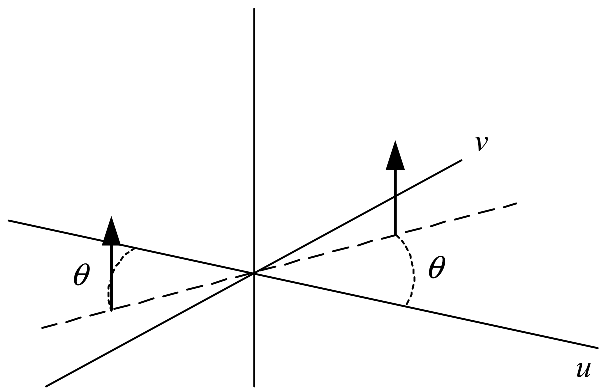
Example





$$\cos(2\pi x) \quad \longleftrightarrow \quad \tfrac{1}{2}\delta(u-1,v)+\tfrac{1}{2}\delta(u+1,v)$$





$$\cos[2\pi(x\cos\theta+y\sin\theta)] \quad \longleftrightarrow \quad \begin{array}{l}\tfrac{1}{2}\delta(u-\cos\theta,v-\sin\theta)+\\ \quad \tfrac{1}{2}\delta(u+\cos\theta,v+\sin\theta)\end{array}$$