3 – IMAGE TRANSFORMS (B)

THE DISCRETE FOURIER TRANSFORM

Suppose that a continuous function $f(x)$ is discretized into a sequence

$$\{f(x_0), f(x_0 + \Delta x), f(x_0 + 2\Delta x), \ldots, f(x_0 + [N-1]\Delta x)\}$$

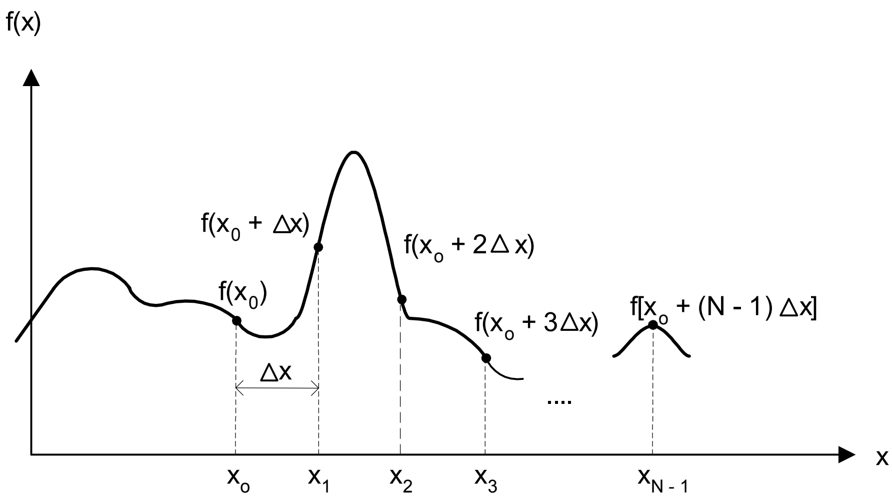
by taking N samples Δx units apart.

Note that x may be used as either a discrete or continuous variable, depending on context. In the discrete case, we define

$$f(x) = f(x_0 + x\Delta x) \qquad x = 0, 1, 2, \ldots, N-1$$

Likewise, in the frequency domain, we have

$$F(u) = F(u\Delta u) \qquad u = 0, 1, 2, \ldots, N-1$$



The discrete Fourier transform (DFT) pair is given by

$$F(u) = \frac{1}{N} \sum_{x=0}^{N-1} f(x) \exp[-j2\pi ux/N]; \quad u = 0, 1, 2, \ldots, N-1 \tag{1}$$

and

$$f(x) = \sum_{u=0}^{N-1} F(u) \exp[j2\pi ux/N]; \quad x = 0, 1, 2, \ldots, N-1 \tag{2}$$

The values $u = 0, 1, 2, \ldots, N-1$ in the DFT correspond to samples of the continuous transform at values $0, \Delta u, 2\Delta u, (N-1)\Delta u$. The terms Δu and Δx are related by

$$\Delta u = \frac{1}{N\Delta x} \tag{3}$$

In the 2D case, the DFT pair is

$$F(u,v) = \frac{1}{MN} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y) \exp[-j2\pi(ux/M + vy/N)]; \tag{4}$$

$$u = 0, 1, 2, \ldots, M-1,\ v = 0, 1, 2, \ldots, N-1,$$

and

$$f(x,y) = \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} F(u,v) \exp[j2\pi(ux/M + vy/N)]; \tag{5}$$

$$\text{for } x = 0, 1, 2, \ldots, M-1,\ y = 0, 1, 2, \ldots, N-1.$$

The sampling increments in the spatial and frequency domains are related by

$$\Delta u = \frac{1}{M\Delta x}, \qquad \Delta v = \frac{1}{N\Delta y} \tag{6}$$

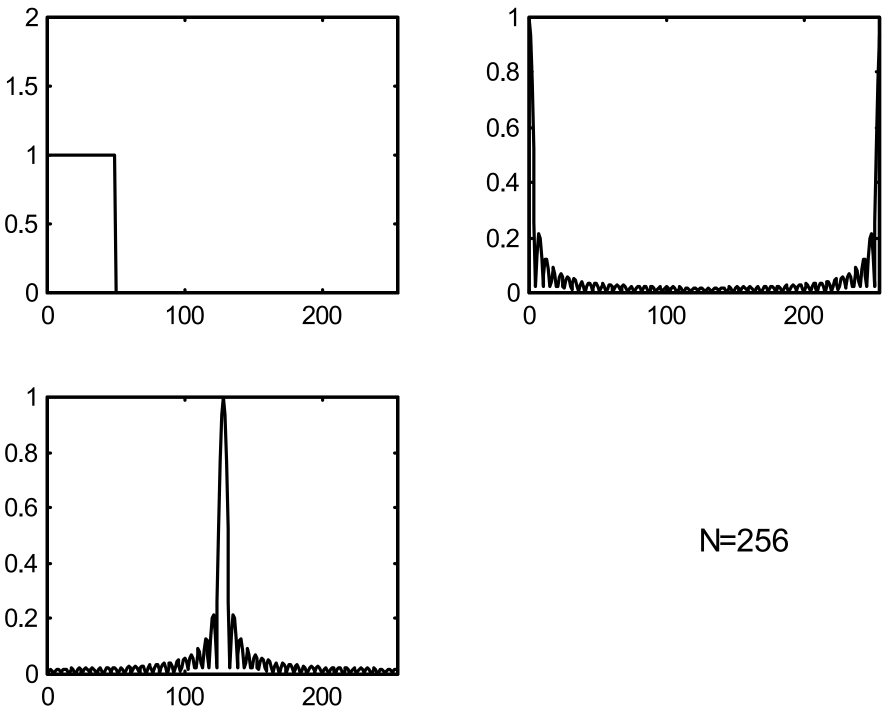
The Fourier spectrum, phase, and energy spectrum of 1D and 2D discrete functions are computed as for the continuous case.

$$\begin{array}{lll} \text{Fourier spectrum:} & |F(u,v)| & = [R^2(u,v) + I^2(u,v)]^{1/2} \\ \text{Phase spectrum:} & \phi(u,v) & = \tan^{-1}[I(u,v)/R(u,v)] \\ \text{Power spectrum:} & P(u,v) & = |F(u,v)|^2 = R^2(u,v) + I^2(u,v) \end{array}$$

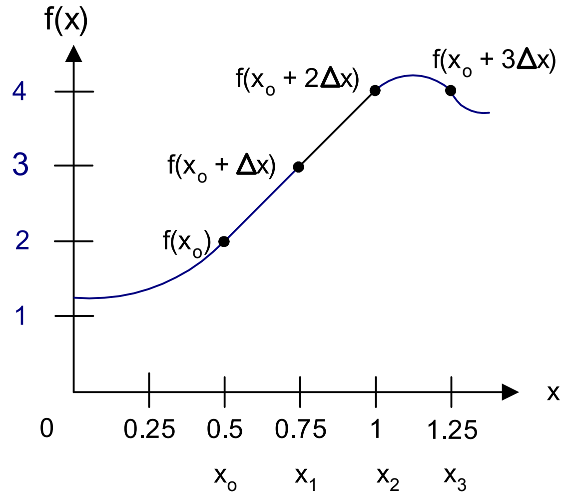
The direct computation of an N-point DFT requires of the order of N^2 operations. For an $M \times N$ array, M^2N^2 operations are required. This can be considerably reduced by the fast Fourier transform (FFT) algorithm to $MN \log_2 M \log_2 N$ operations. Suppose $M = N = 2^9$. Then

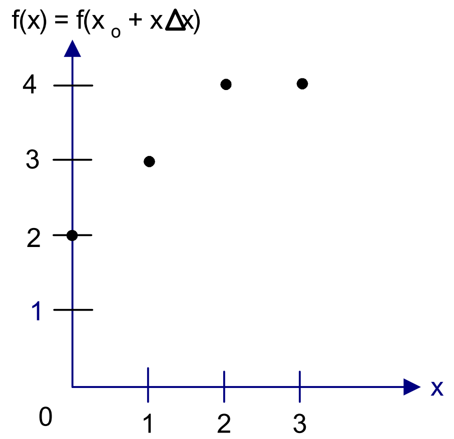
$$\begin{aligned} \text{Direct DFT:} \quad & M^2N^2 &= 69 \times 10^9 \\ \text{FFT:} \quad & MN \log_2 M \ \log_2 N &= 21 \times 10^6 \end{aligned}$$

Example (1D DFT)



Example (1D DFT)





Sampling takes place at: $x_0 = 0.5,\ x_1 = 0.75,\ x_2 = 1.0,\ x_2 = 1.25$

$$
\begin{aligned}
F(u) &= \frac{1}{N}\sum_{x=0}^{N-1} f(x)\exp[-j2\pi ux/N] \\
F(0) &= \frac{1}{4}\sum_{x=0}^{3} f(x)\exp(0) \\
&= \frac{1}{4}[f(0)+f(1)+f(2)+f(3)] \\
&= \frac{1}{4}[2+3+4+4] \\
&= 3.25 \\
F(1) &= \frac{1}{4}\sum_{x=0}^{3} f(x)\exp[-j2\pi x/4] \\
&= \frac{1}{4}[f(0)\exp(0)+f(1)\exp(-j\pi/2)+f(2)\exp(-j\pi)+f(3)\exp(-j2\pi)] \\
&= \frac{1}{4}(-2+j)
\end{aligned}
$$

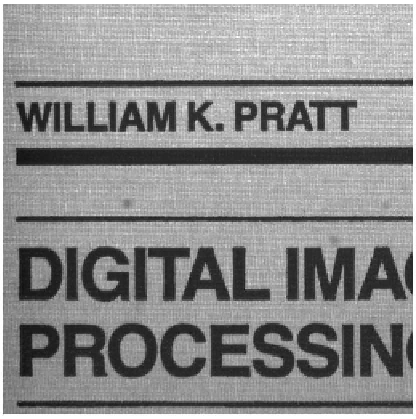
Similarly, $F(2) = -\frac{1}{4}(1 + j\times 0)$, $F(3) = -\frac{1}{4}(2+j)$

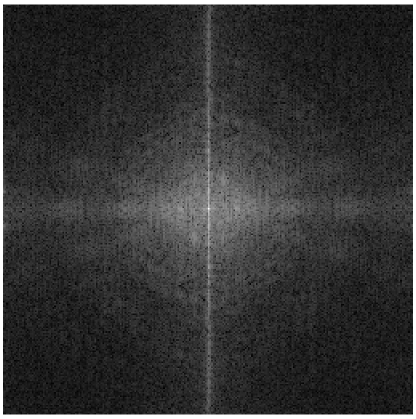
Fourier spectrum:

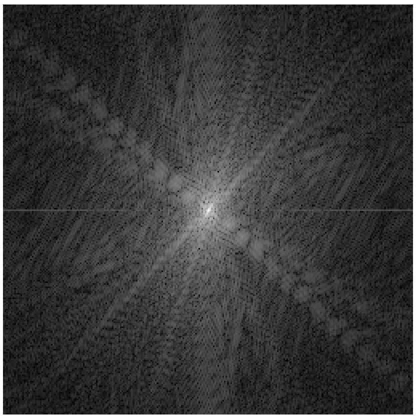
$$|F(0)| = 3.25, \quad |F(1)| = 0.56 \quad |F(2)| = 0.25 \quad |F(3)| = 0.56$$

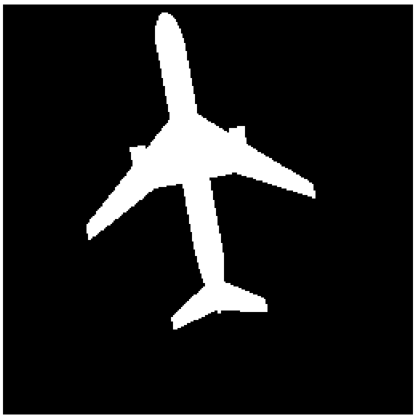
In this example,
$N = 4$, $\Delta x = 0.25$, $\Delta u = 1/(N\Delta x) = 1$

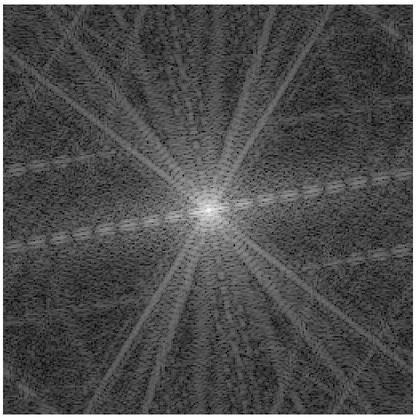
Examples of 2D DFT











Some Properties of the 2D DFT

Separability

We can write Eq. (4) in the separable form

$$F(u,v) = \frac{1}{MN}\sum_{x=0}^{M-1}\exp[-j2\pi ux/M]\sum_{y=0}^{N-1}f(x,y)\exp[-j2\pi vy/N] \quad (7)$$

$$= \frac{1}{M}\sum_{x=0}^{M-1}F(x,v)\exp[-j2\pi ux/M] \quad (8)$$

where

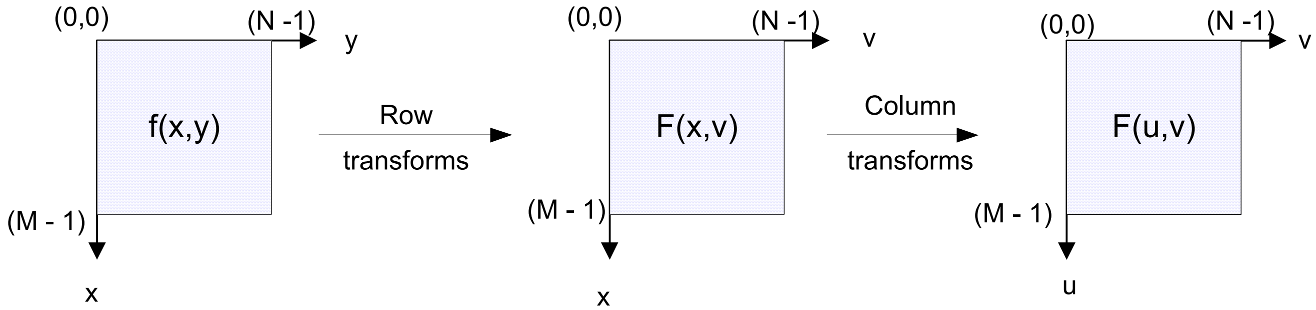
$$F(x,v) = \left[\frac{1}{N}\sum_{y=0}^{N-1}f(x,y)\exp[-j2\pi vy/N]\right] \quad (9)$$

for $u = 0, 1, \ldots, M-1$, $v = 0, 1, \ldots, N-1$,

The principal advantage of the separability property is that $F(u,v)$ (or $f(x,y)$) can be obtained in two steps by successive applications of the 1D Fourier transform (or its inverse).

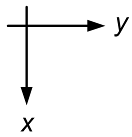
For each value of x, the expression inside the brackets is a 1D transform, with frequency values $v = 0, 1, \ldots, N-1$. Therefore the 2D function $F(x,v)$ is obtained by taking a transform along each row of $f(x,y)$. The desired result $F(u,v)$ is then obtained by taking a transform along each column of $F(x,v)$.

The same results may be obtained by first taking transforms along the columns of $f(x,y)$ and then along the rows of that result.



Example

(In this example, the x axis points down, the y axis to the right.)



$$f(x,y) = \begin{bmatrix} 4 & 2 & 2 & 1 \\ 2 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Hence, we have

$$\begin{aligned} F(x,v) &= \frac{1}{4}\begin{bmatrix} 9 & 2-j & 3 & 2+j \\ 4 & 1-j & 2 & 1+j \\ 2 & 1-j & 0 & 1+j \\ 1 & 1 & 1 & 1 \end{bmatrix} \\ F(u,v) &= \frac{1}{16}\begin{bmatrix} 16 & 5-j3 & 6 & 5+j3 \\ 7-j3 & 0 & 3-j1 & 2 \\ 6 & 1-j1 & 0 & 1+j \\ 7+j3 & 2 & 3+j & 0 \end{bmatrix} \end{aligned}$$

Translation

The translation properties are

$$f(x - x_0, y - y_0) \Leftrightarrow F(u, v)\exp[-j2\pi(ux_0/M + vy_0/N)] \tag{10}$$

and

$$f(x, y)\exp[j2\pi(u_0x/M + v_0y/N)] \Leftrightarrow F(u - u_0, v - v_0) \tag{11}$$

Note that a shift in $f(x, y)$ does not affect the magnitude of its Fourier transform since

$$|F(u, v)\exp[-j2\pi(ux_0/M + vy_0/N)]| = |F(u, v)| \tag{12}$$

Average Value

The average value of a 2D discrete function is

$$\bar{f}(x, y) = \frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x, y) \tag{13}$$

It is easily shown that

$$\bar{f}(x, y) = F(0, 0) \tag{14}$$

Periodicity and Conjugate Symmetry

The DFT and its inverse are periodic with period N.

For the 1D case,

$$F(u) = F(u + N); \qquad f(x) = f(x + N)$$

Furthermore, if $f(x)$ is real, the DFT also exhibits conjugate symmetry:

$$F(u) = F^*(-u)$$

or

$$|F(u)| = |F(-u)|$$

⇒ The periodicity property indicates that $F(u)$ has a period of length. N

⇒ The symmetry property shows that the magnitude of the transform is centered on the origin.

Hence,

- The magnitudes of the transform values from $(N/2)+1$ to $N-1$ are reflections of the values in the half period to the left of the origin.
- The magnitudes of the transform values are symmetrical about $u = N/2$.

To move the origin of the transform to the point $u = v = N/2$, we multiply $f(x, y)$ by $(-1)^{x+y}$ prior to taking the transform.

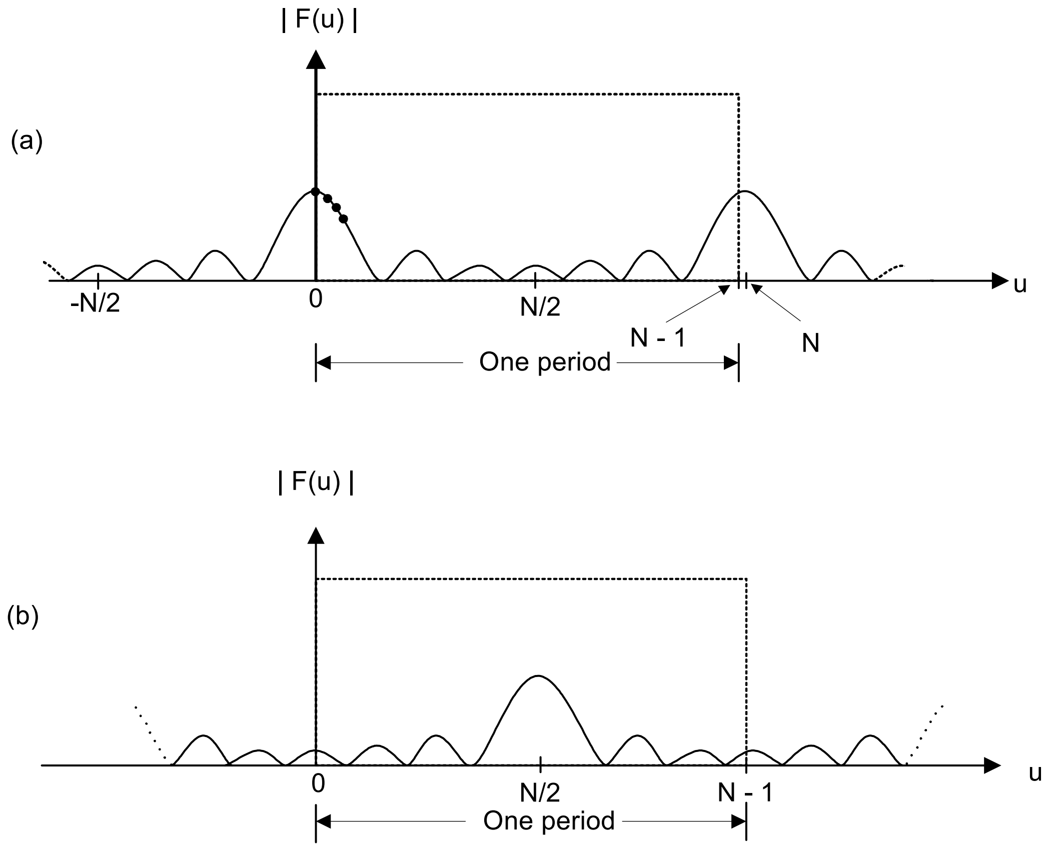
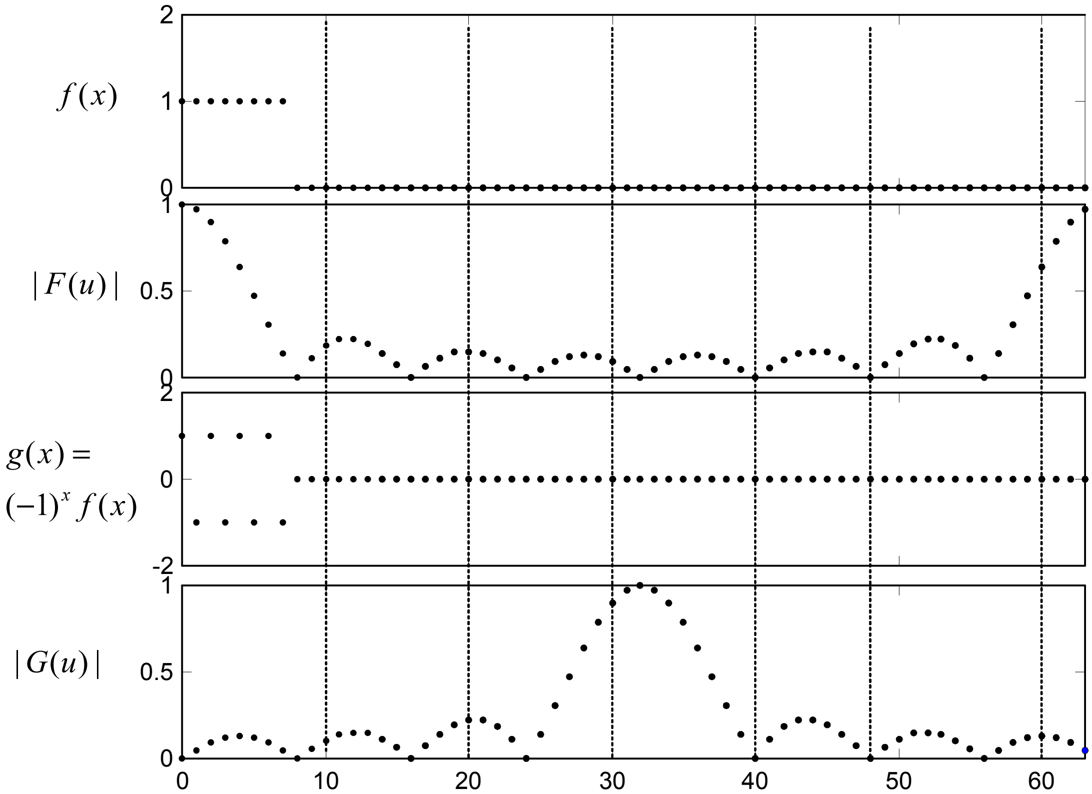


Illustration of the periodicity properties of the discrete Fourier transform:
(a) Fourier spectrum showing back-to-back half periods in the interval [0, N - 1];
(b) Shifted spectrum showing a full period in the same interval.

Proof of Periodicity and Conjugate Symmetry

$$\begin{aligned}
F(u) &= (1/N)\sum_{x=0}^{N-1} f(x)\exp[-j2\pi ux/N] \\
F(u+N) &= (1/N)\sum_{x} f(x)\exp[-j2\pi(u+N)x/N] \\
&= (1/N)\sum_{x} f(x)\exp[-j2\pi ux/N - j2\pi Nx/N] \\
&= (1/N)\sum_{x} f(x)\exp[-j2\pi ux/N]\exp[-j2\pi x] \\
&= (1/N)\sum_{x} f(x)\exp[-j2\pi ux/N] \quad \text{since } \exp[-j2\pi x] \equiv 1 \\
&= F(u)
\end{aligned}$$

$$\begin{aligned}
F(-u) &= (1/N)\sum_{x} f(x)\exp[j2\pi ux/N] \\
F^*(-u) &= (1/N)\sum_{x} f^*(x)\exp[-j2\pi ux/N] \\
&= (1/N)\sum_{x} f(x)\exp[-j2\pi ux/N] \quad \text{for real } f \\
&= F(u)
\end{aligned}$$



For the 2D case involving an $N \times N$ image,

$$F(u,v) = F(u+N,v) = F(u,v+N) = F(u+N,v+N) \tag{15}$$

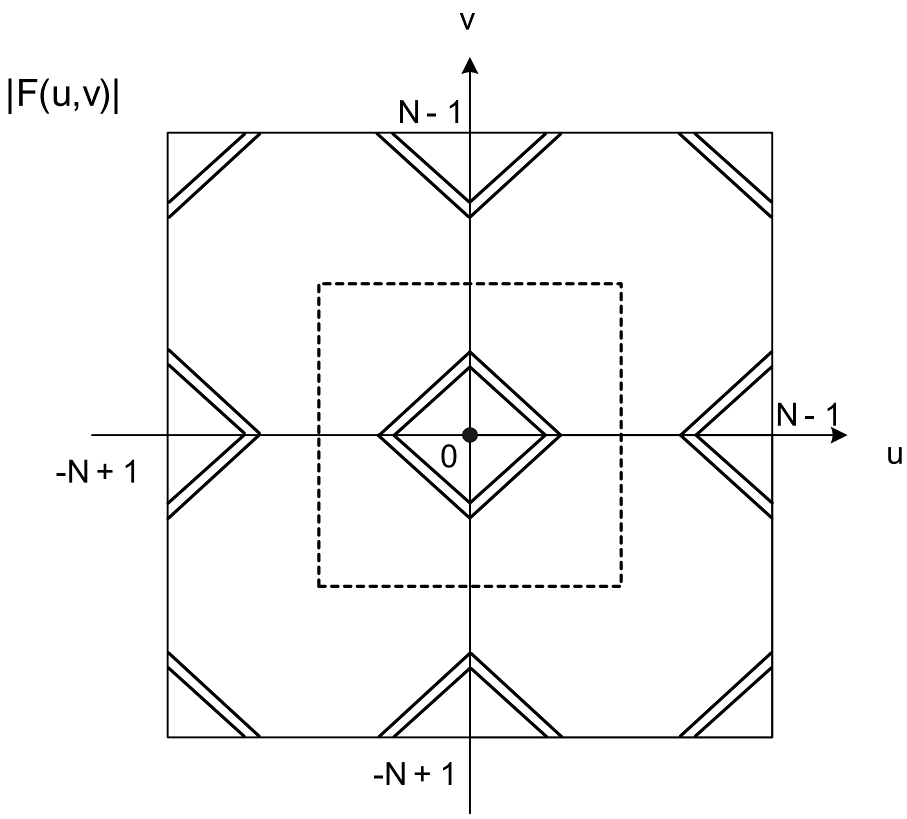
Although $F(u,v)$ repeats itself for infinitely many values of u and v, only the N values of each variable in any one period are required to obtain $f(x,y)$ from $F(u,v)$. Similar comments apply to $f(x,y)$ in the spatial domain.

If $f(x,y)$ is real, the DFT also exhibits conjugate symmetry:

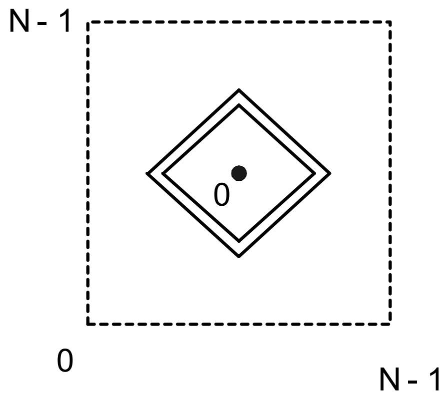
$$F(u,v) = F^*(-u,-v) \tag{16}$$

or

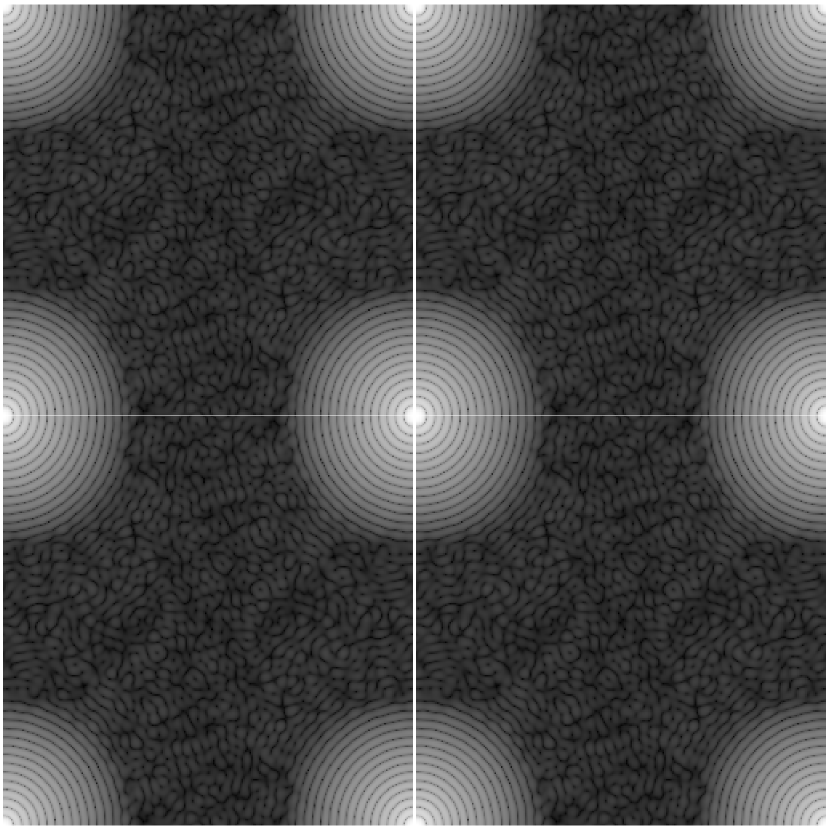
$$|F(u,v)| = |F(-u,-v)| \tag{17}$$

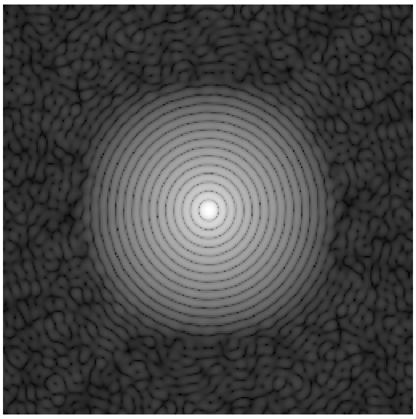


v
|F(u,v)|
N - 1
N - 1
u
-N + 1
0
-N + 1



N - 1
0
0
N - 1





Example

Consider a 100×100 image $f(x,y)$ given by

$$f(x,y) = \begin{cases} 100 & 0 \le x \le 1, \quad 0 \le y \le 2 \\ 0 & \text{elsewhere} \end{cases}$$

$x, y = 0, 1, \ldots, 99$. Determine the points along the u axis where $F(u,v) = 0$ and sketch $|F(u,0)|$ for $0 \le u \le 99$.

We have

$$F(u,v) = \frac{1}{N^2} \sum_{x=0}^{N-1} \sum_{y=0}^{N-1} f(x,y) \exp[-j2\pi(ux/N + vy/N)]$$

where $N = 100$. Let

$$\begin{aligned}
G(u) &= F(u,0) \\
&= \frac{1}{N^2} \sum_{x=0}^{N-1} \sum_{y=0}^{N-1} f(x,y) \exp[-j2\pi(ux/N)] \\
&= \begin{array}{ll} (1/N^2)[f(0,0)\exp(0) & +f(1,0)\exp(-j2\pi u/N) \\ \quad +f(0,1)\exp(0) & +f(1,1)\exp(-j2\pi u/N) \\ \quad +f(0,2)\exp(0) & +f(1,2)\exp(-j2\pi u/N) \\ \quad +0] & \end{array} \\
&= (1/N^2)[300 + 300\exp(-j2\pi u/N)] \\
&= (300/N^2)[1 + \exp(-j2\pi u/N)]
\end{aligned}$$

$$\begin{aligned}
G(u) = 0 \quad &\Rightarrow \quad 1 + \exp(-j2\pi u/N) = 0 \\
&\Rightarrow \quad 1 + \cos(2\pi u/N) - j\sin(2\pi u/N) = 0 \\
&\Rightarrow \quad u = N/2 = 50
\end{aligned}$$

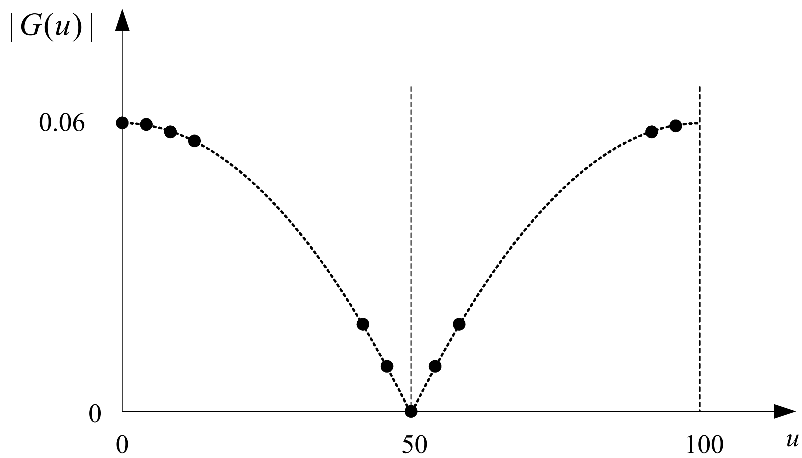
From above,

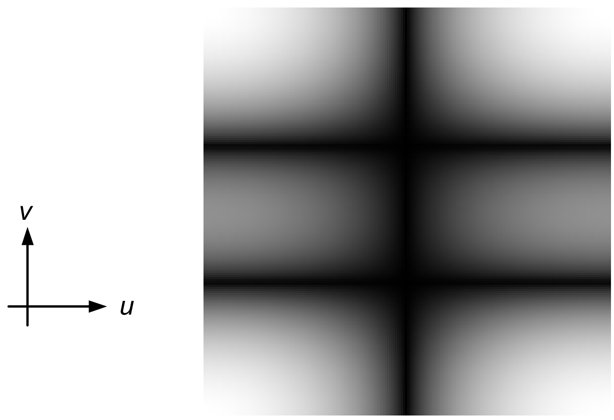
$$
\begin{aligned}
G(u) &= (300/N^2)[1+\exp(-j2\pi u/N)] \\
&= (300/N^2)\exp(-j\pi u/N)[\exp(j\pi u/N)+\exp(-j\pi u/N)] \\
&= (300/N^2)\times 2\cos(\pi u/N)\exp(-j\pi u/N)
\end{aligned}
$$

$$
\begin{aligned}
|G(u)| &= (300/N^2)\times 2|\cos(\pi u/N)| \\
&= 0.06|\cos(\pi u/N)|
\end{aligned}
$$

We note the following:

- $|G(u)|_{max} = 0.06$ at $u = 0$
- $|G(u)|_{min} = 0$ at $u = N/2 = 50$
- $|G(u)|$ decreases monotonically as u increases from $u = 0$ to $u = 50$
- $|G(u)|$ is symmetrical about $u = 50$





Convolution

1D Continuous Case

The convolution of two functions $f(x)$ and $g(x)$ is defined by

$$f(x) \star g(x) = \int_{-\infty}^{\infty} f(\alpha) g(x - \alpha) d\alpha \tag{18}$$

It can be shown that

$$f(x) \star g(x) \Leftrightarrow F(u)G(u) \tag{19}$$

$$f(x)g(x) \Leftrightarrow F(u) \star G(u) \tag{20}$$

i.e., convolution in the x domain in equivalent to multiplication in the u domain, and vice versa. These two results are known as the convolution theorem.

1D Discrete Case

Suppose that $f(x)$ and $g(x)$ are discretized into sampled arrays of size A and B, respectively:

$$\{f(0), f(1), f(2), \ldots, f(A-1)\}$$

and

$$\{g(0), g(1), g(2), \ldots, g(B-1)\}$$

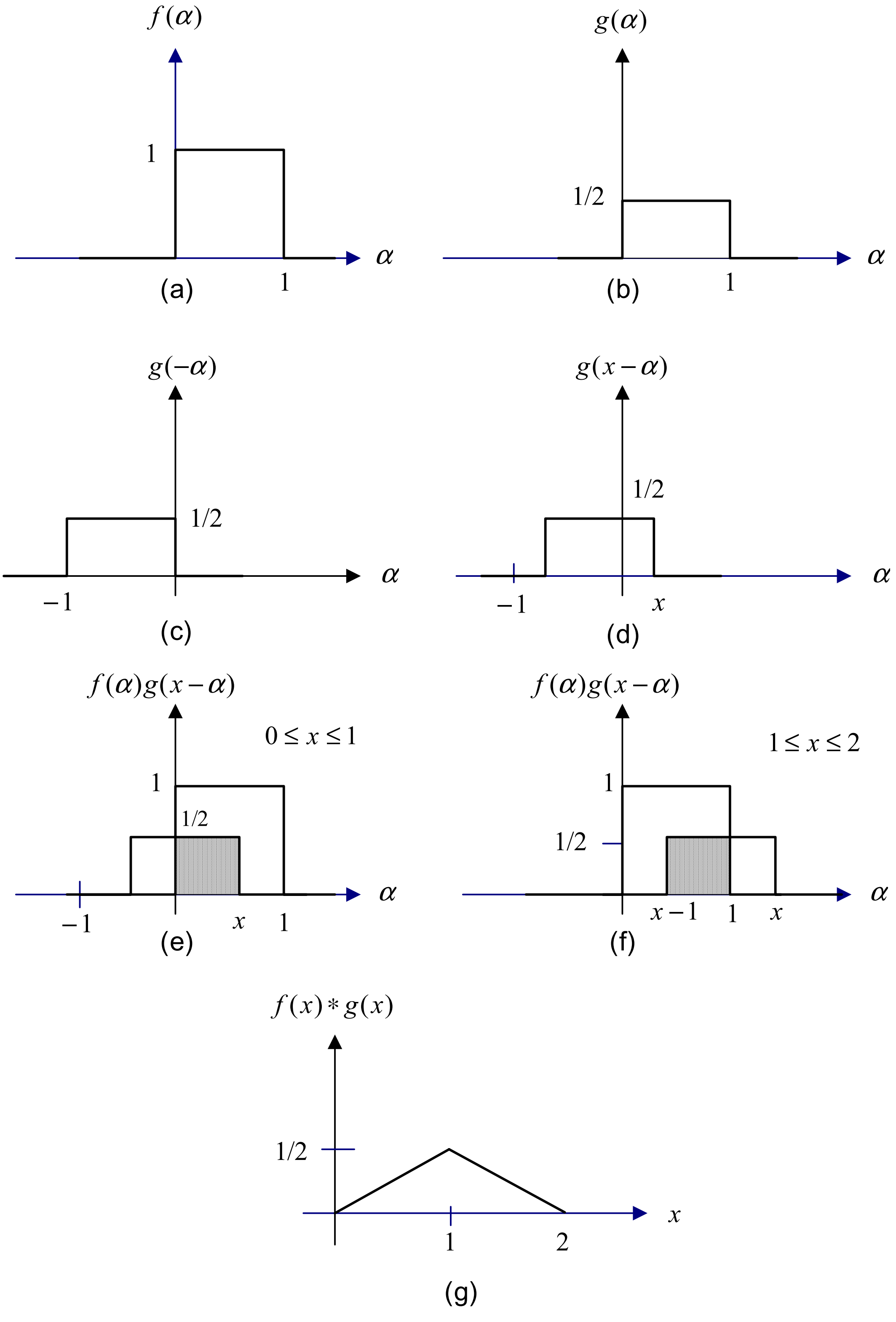
The discrete convolution of $f(x)$ and $g(x)$ is given by

$$h(x) = f(x) \star g(x) = \sum_{m=-\infty}^{+\infty} f(m) g(x - m) \tag{21}$$

i.e.,

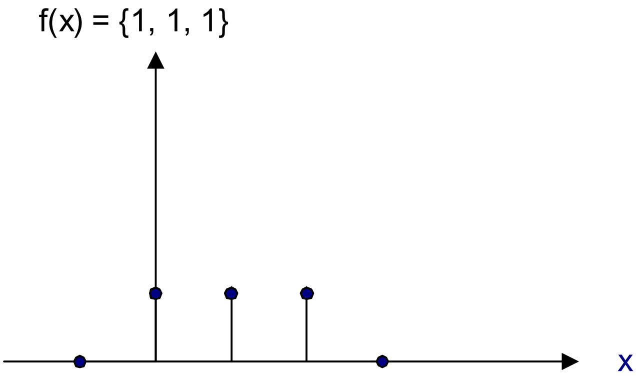
$$\begin{aligned} h(0) &= \sum_{m=-\infty}^{+\infty} f(m) g(-m) \\ h(1) &= \sum_{m=-\infty}^{+\infty} f(m) g(1 - m) \\ &\textit{etc.} \end{aligned}$$

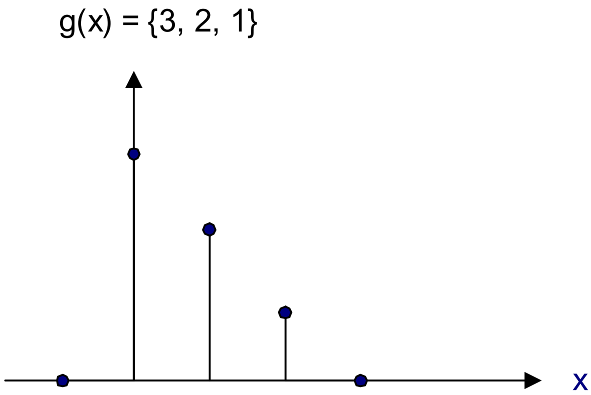
The convolution of these two arrays will result in a sequence of length $A + B - 1$.



Graphic illustration of convolution.
The shaded areas indicate regions where the product is not zero.

Example

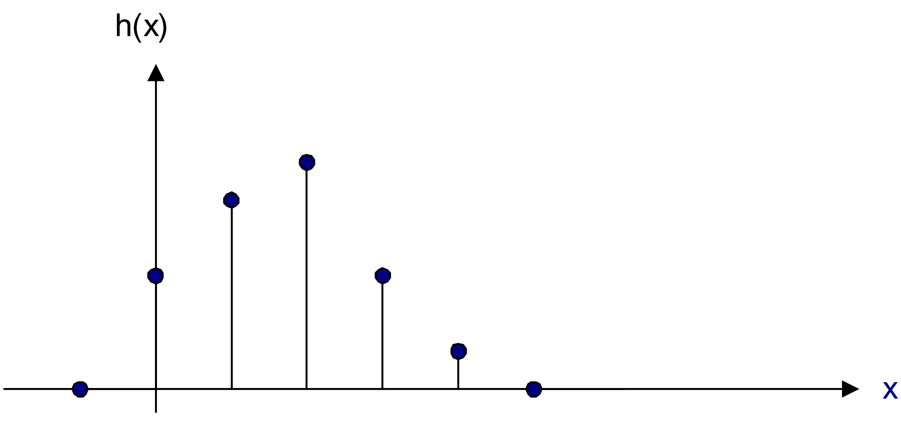




$$
\begin{array}{lcl}
x < 0 & : & h(x) = 0 \\
x = 0 & : & h(0) = 1 \times 3 = 3 \\
x = 1 & : & h(1) = 1 \times 2 + 1 \times 3 = 5 \\
x = 2 & : & h(2) = 1 \times 1 + 1 \times 2 + 1 \times 3 = 6 \\
x = 3 & : & h(3) = 1 \times 1 + 1 \times 2 = 3 \\
x = 4 & : & h(4) = 1 \times 1 = 1 \\
x > 4 & : & h(x) = 0
\end{array}
$$

Hence,

$$h(x) = \{3, 5, 6, 3, 1\}$$



The DFT and its inverse are periodic functions. Formulating a discrete convolution theorem to be consistent with periodicity involves assuming that the discrete function $f(x)$ and $g(x)$ are periodic with some period M. The resulting convolution is then periodic with the same period.

For the discrete convolution theorem to hold, we require

$$M \geq A + B - 1 \tag{22}$$

Zeros are appended to the samples so that each input signal is of length M. The extended sequences are

$$f_e(x) = \begin{cases} f(x) & 0 \leq x \leq A - 1 \\ 0 & A \leq x \leq M - 1 \end{cases} \tag{23}$$

and

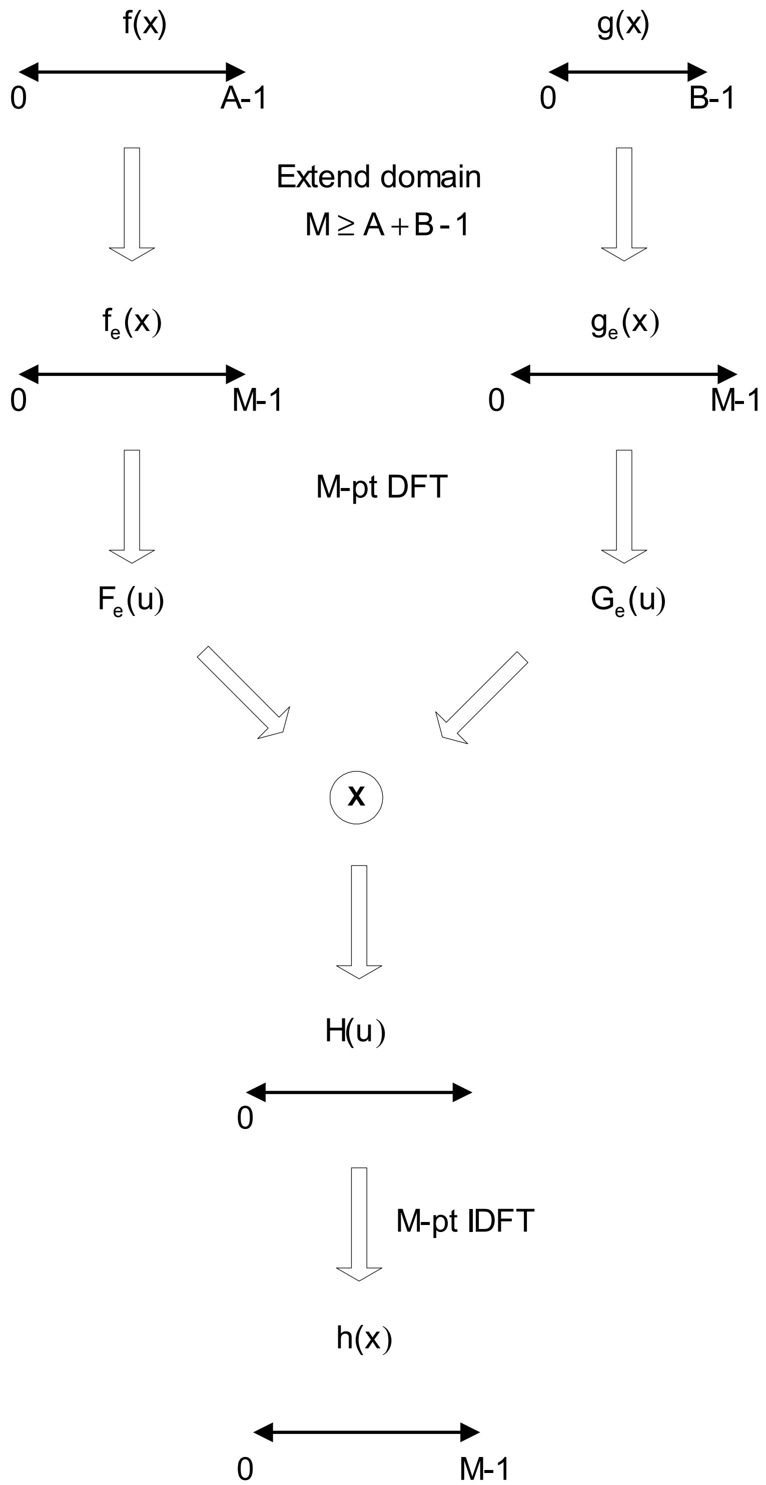
$$g_e(x) = \begin{cases} g(x) & 0 \leq x \leq B - 1 \\ 0 & B \leq x \leq M - 1 \end{cases} \tag{24}$$

The discrete convolution of $f_e(x)$ and $g_e(x)$ is

$$f_e(x) \star g_e(x) = \frac{1}{M} \sum_{m=0}^{M-1} f_e(m) g_e(x - m); \quad x = 0, 1, 2, \ldots, M - 1 \tag{25}$$

The convolution function is a discrete, periodic array of length M, with the values $x = 0, 1, 2, \ldots, M-1$ describing a full period of $f_e(x) \star g_e(x)$.

Convolution Theorem: $h(x) = f_e(x) * g_e(x) \Leftrightarrow F_e(u) \times G_e(u)$



2D Continuous Case
In the 2D case,

$$f(x,y) \star g(x,y) = \iint_{-\infty}^{\infty} f(\alpha,\beta) g(x-\alpha, y-\beta) d\alpha \, d\beta \tag{26}$$

The convolution theorem is

$$\begin{aligned} f(x,y) \star g(x,y) &\Leftrightarrow F(u,v)G(u,v) && (27) \\ f(x,y)g(x,y) &\Leftrightarrow F(u,v) \star G(u,v) && (28) \end{aligned}$$

2D Discrete Case
The 2D, discrete convolution is formulated by letting $f(x,y)$ and $g(x,y)$ be discrete arrays of size $A \times B$ and $C \times D$, respectively. These arrays must be assumed periodic with some period M and N in the x and y directions, respectively. Wraparound error in the individual convolution periods is avoided by choosing

$$\begin{aligned} M &\geq A + C - 1 \\ N &\geq B + D - 1 \end{aligned}$$

The periodic sequences are formed by extending $f(x,y)$ and $g(x,y)$:

$$f_e(x,y) = \begin{cases} f(x,y) & 0 \leq x \leq A-1 \quad \text{and} \quad 0 \leq y \leq B-1 \\ 0 & A \leq x \leq M-1 \quad \text{or} \quad B \leq y \leq N-1 \end{cases} \tag{29}$$

and

$$g_e(x,y) = \begin{cases} g(x,y) & 0 \leq x \leq C-1 \quad \text{and} \quad 0 \leq y \leq D-1 \\ 0 & C \leq x \leq M-1 \quad \text{or} \quad D \leq y \leq N-1 \end{cases} \tag{30}$$

The 2D convolution of $f_e(x,y)$ and $g_e(x,y)$ is defined by the relation

$$f_e(x,y) \star g_e(x,y) = \frac{1}{MN} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f_e(m,n) g_e(x-m, y-n) \tag{31}$$

for $x = 0, 1, 2, \ldots, M-1$ and $y = 0, 1, 2, \ldots, N-1$. The $M \times N$ array is one period of the discrete 2D convolution.

The continuous convolution theorem also applies to the discrete case.

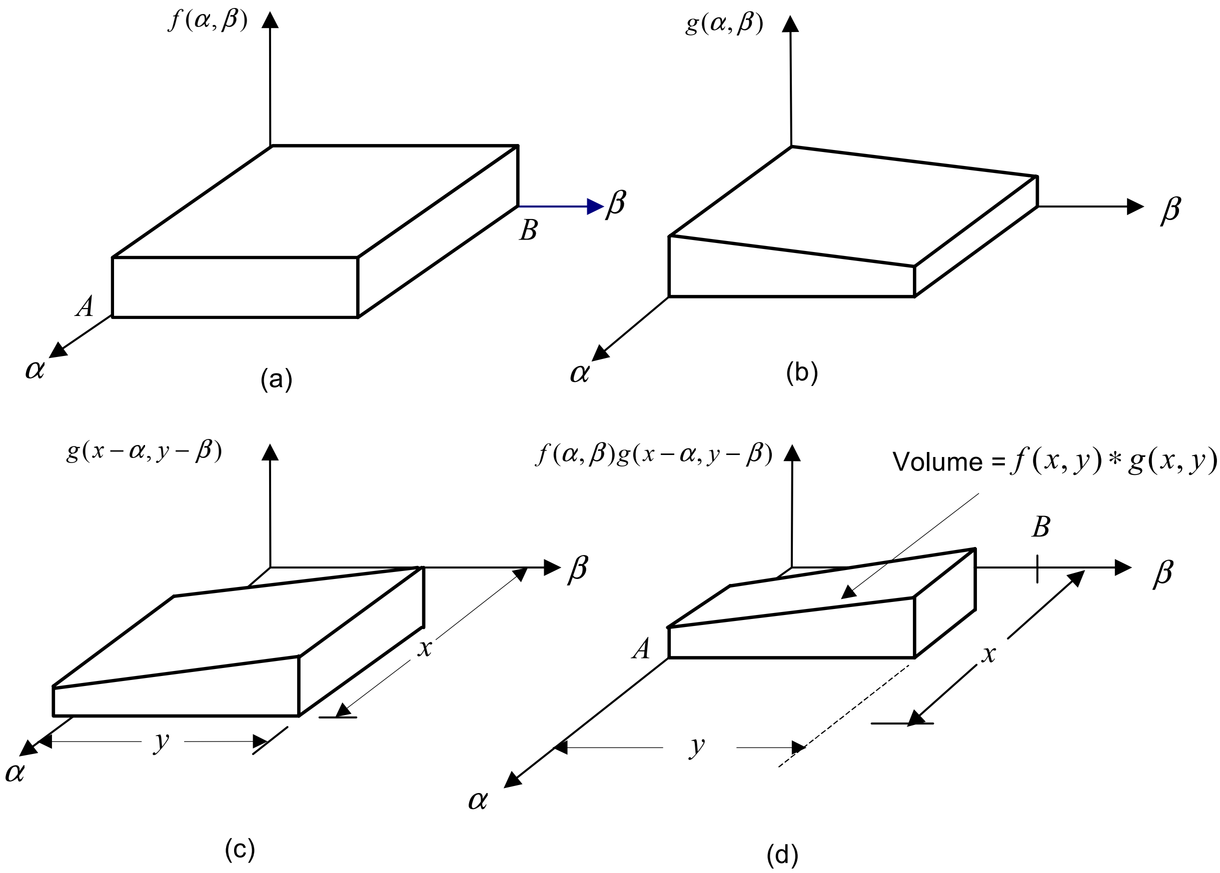


Illustration of the folding, displacement, and multiplication steps needed to perform two dimensional convolution.

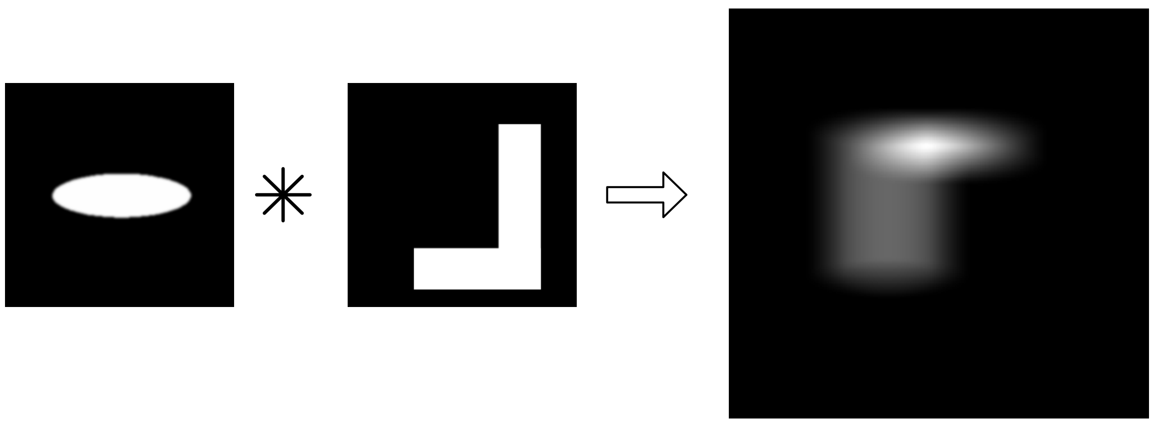


IMAGE SAMPLING

In digital image processing systems, one deals with arrays of numbers obtained by spatially sampling points of a physical image. Image samples nominally represent some physical measurements of a continuous image field, e.g., measurements of the image intensity or photographic density.

Given $f(x, y)$, which denotes a continuous, infinite-dimensional ideal image field representing the intensity of a physical image, the sampled image is given by

$$f_s(x, y) = f(x, y)s(x, y) \tag{32}$$

where

$$s(x, y) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \delta(x - m\Delta x, y - n\Delta y) \tag{33}$$

Hence

$$f_s(x, y) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} f(m\Delta x, n\Delta y)\delta(x - m\Delta x, y - n\Delta y) \tag{34}$$

where $f_s(x, y)$ is evaluated only at sample points $(m\Delta x, n\Delta y)$.

The Fourier transform of $f_s(x, y)$ is

$$F_s(u, v) = \int\int_{-\infty}^{+\infty} f_s(x, y) \exp[-j(2\pi(ux + vy)]dxdy \tag{35}$$

or, equivalently, by the convolution theorem,

$$F_s(u, v) = F(u, v) \star S(u, v) \tag{36}$$

where $S(u, v)$ is the Fourier transform of $s(x, y)$.

It can be shown that

$$S(u,v) = \frac{1}{\Delta x \Delta y} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \delta(u - mu_s, v - nv_s)$$

where

$$u_s = \frac{1}{\Delta x}, \qquad v_s = \frac{1}{\Delta y}$$

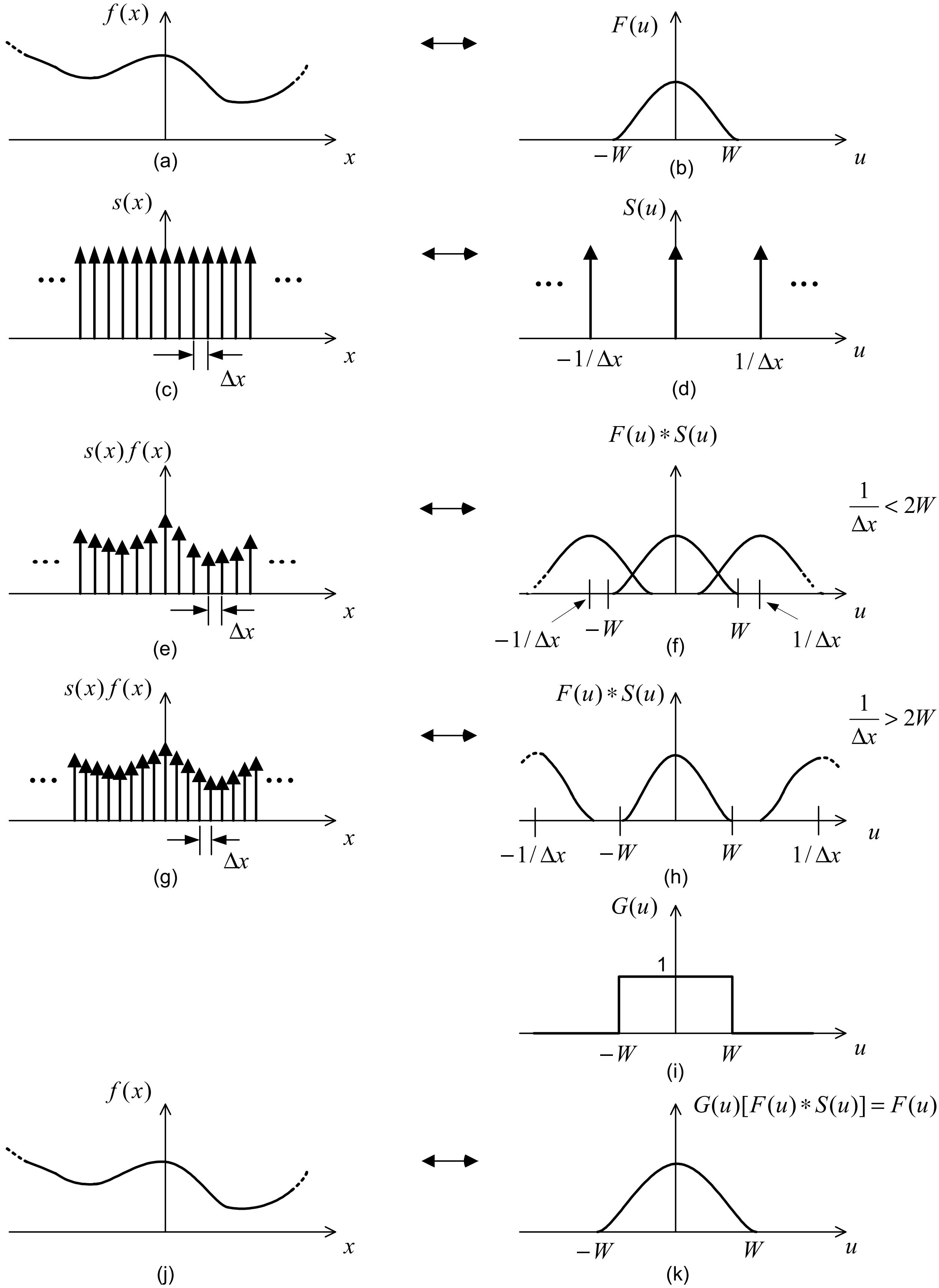
Hence, the convolution of $F(u,v)$ and $S(u,v)$ is

$$F_s(u,v) = \frac{1}{\Delta x \Delta y} \int\int_{-\infty}^{+\infty} \left[F(u-\alpha, v-\beta) \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \delta(\alpha - mu_s, \beta - nv_s) \right] d\alpha d\beta \tag{37}$$

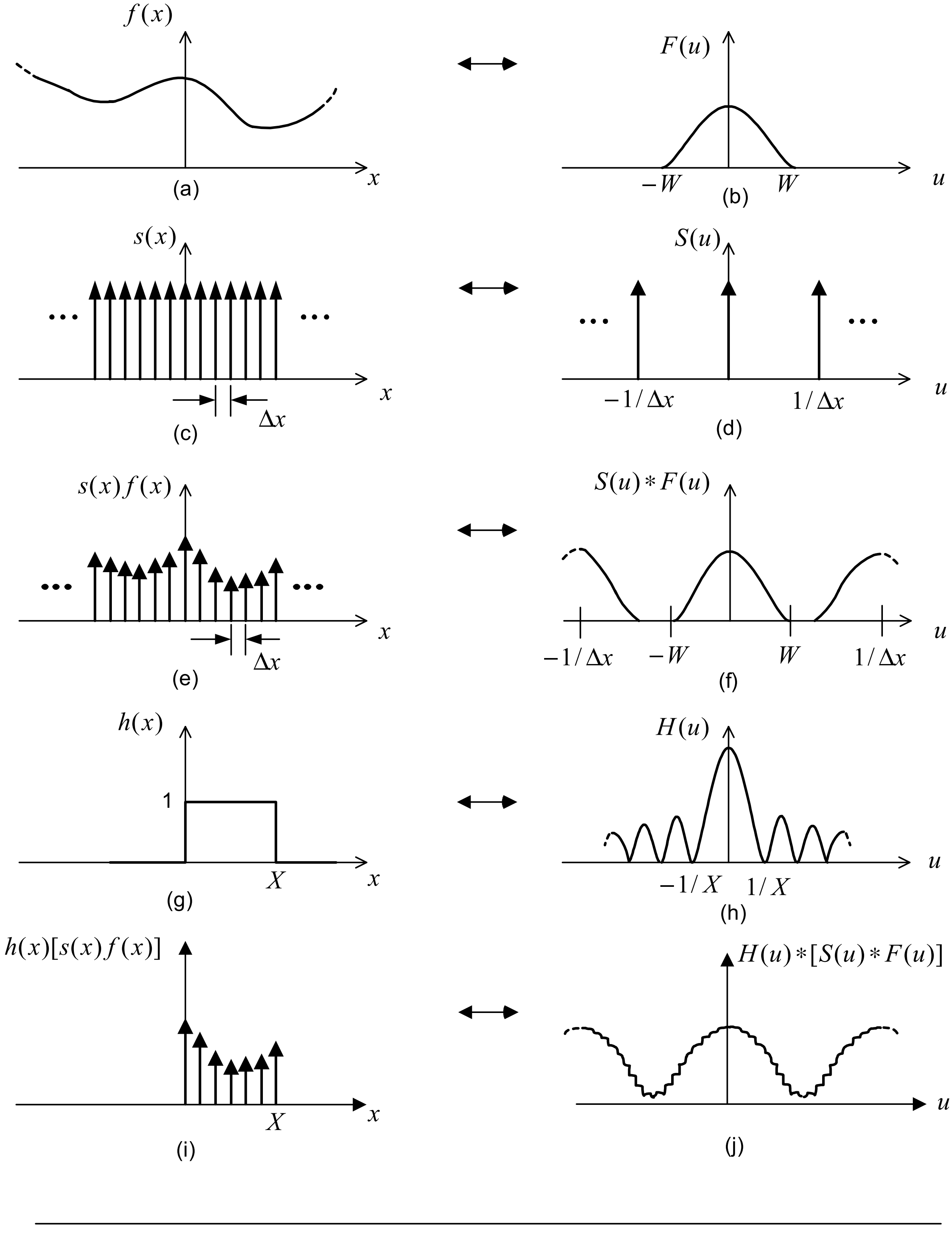
Upon changing the order of summation and integration and invoking the sifting property of the delta function, the sampled image spectrum becomes

$$F_s(u,v) = \frac{1}{\Delta x \Delta y} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} F(u - mu_s, v - nv_s) \tag{38}$$

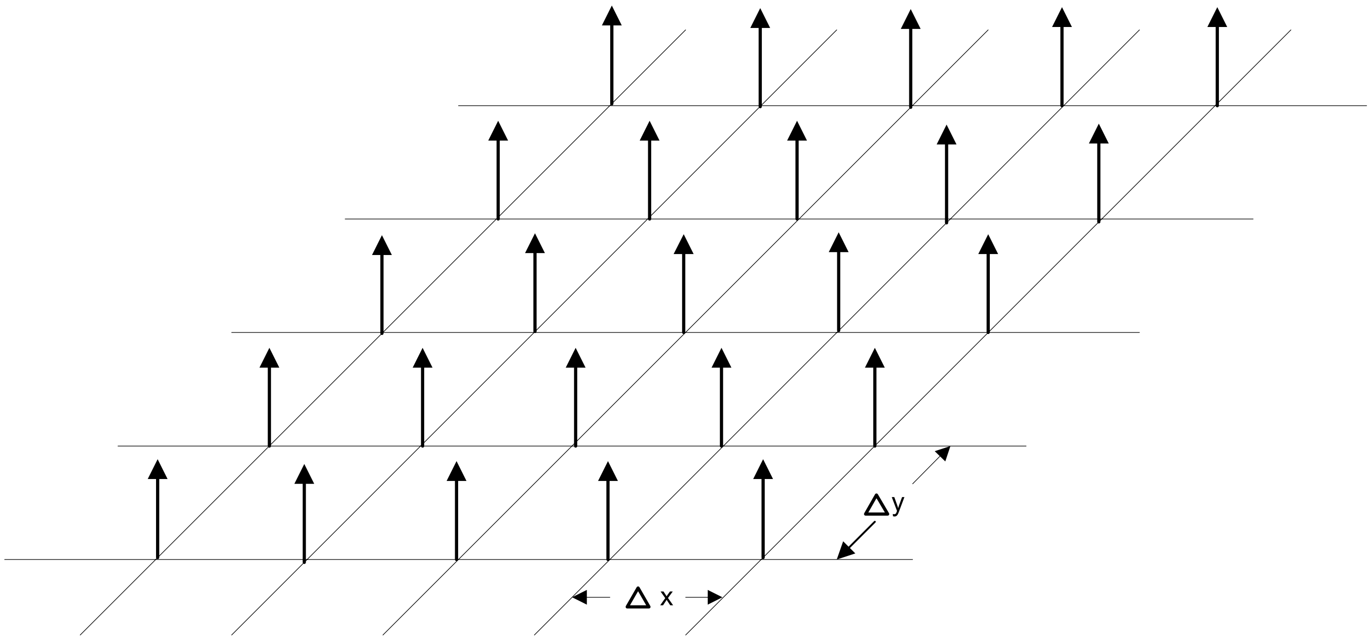
The spectrum of the sampled image consists of the spectrum of the ideal continuous image infinitely repeated over the spatial frequency plane in a grid of resolution $(1/\Delta x, 1/\Delta y)$.



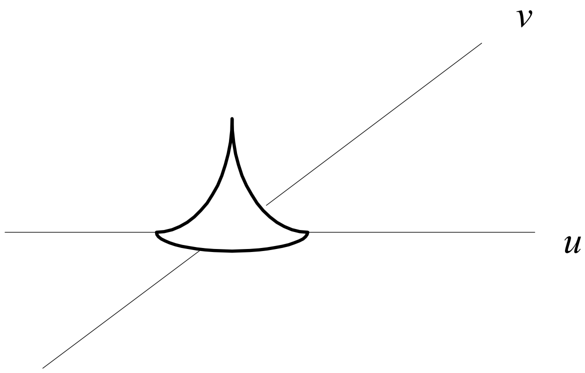
Graphic illustration of sampling concepts



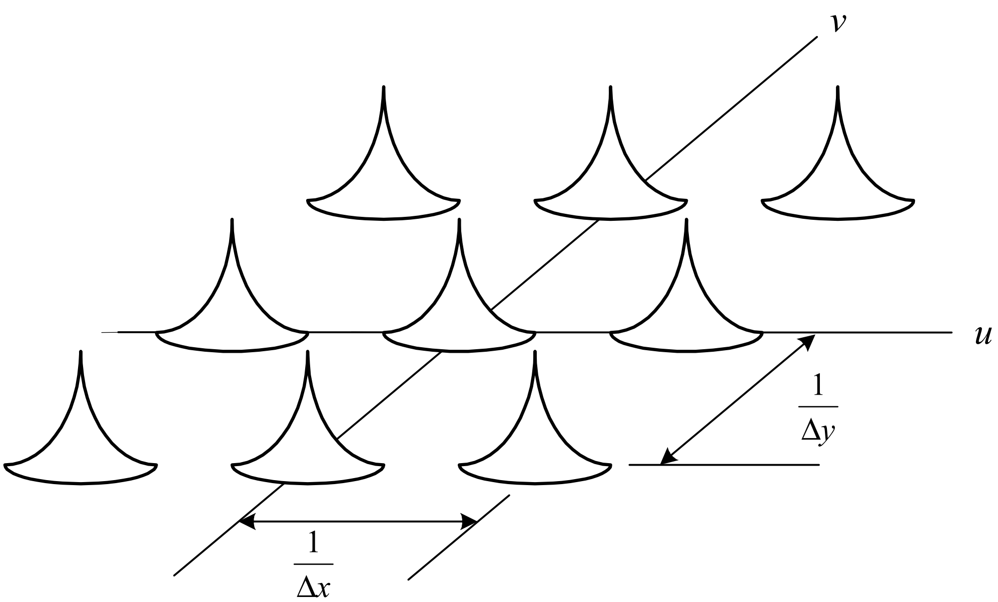
Graphic illustration of finite-sampling concepts



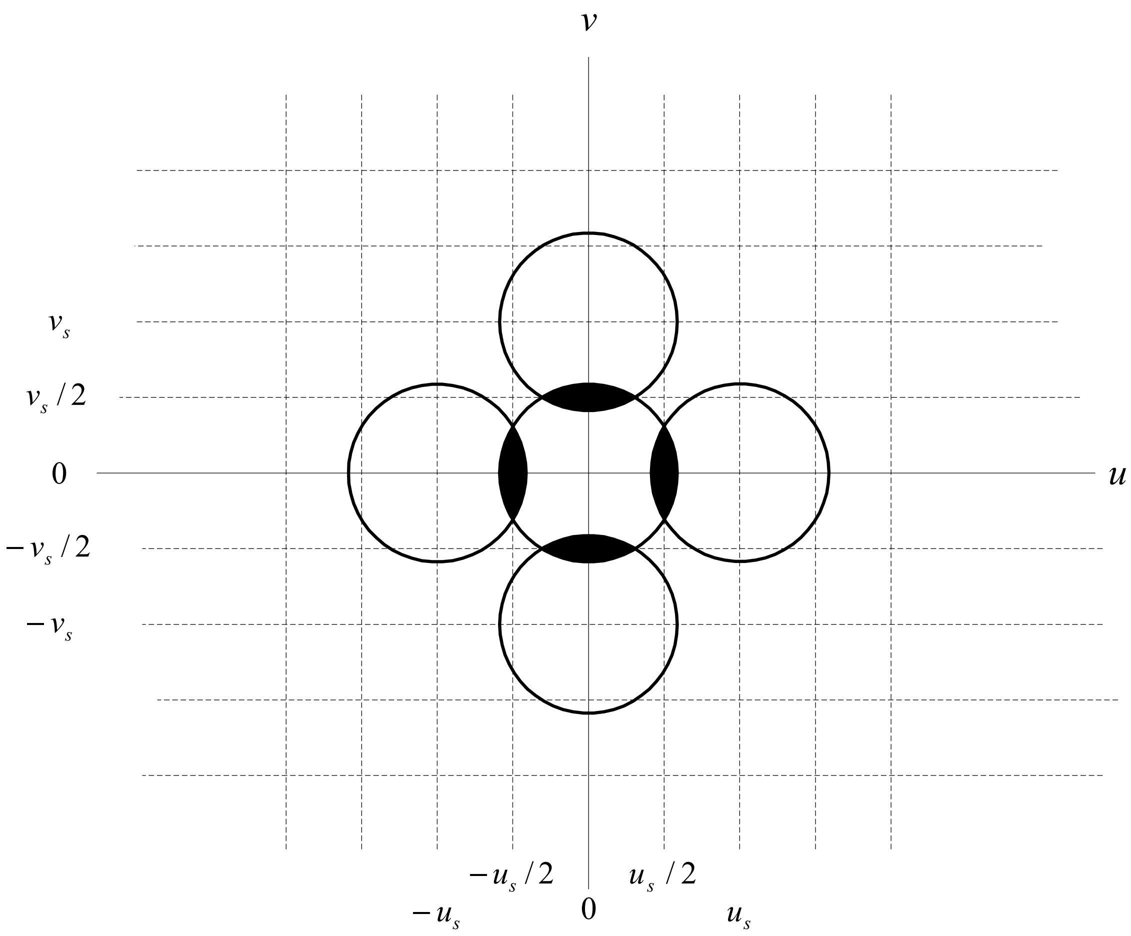
Dirac delta function sampling array.



(a) Spectrum of original image



(b) Spectrum of sampled image



Spectra of undersampled two-dimensional function